THE

OXFORD ENGINEERING SCIENCE SERIES

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General Editors: E. B. Moullin, D. R. Pye, R. V. Southwell

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RELAXATION METHODS IN ENGINEERING SCIENCE

A TREATISE ON APPROXIMATE COMPUTATION

BY

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OXFORD UNIVERSITY PRESS

OXFORD UNIVERSITY PRESS
AMEN HOUSE, E.C. 4
London Edinburgh Glasgow New York
Toronto Melbourne Cape Town Bombay
Calcutta Madras
GEOFFREY CUMBERLEGE
PUBLISHER TO THE UNIVERSITY

FIRST EDITION 1940
Reprinted photographically in Great Britain in 1943, 1946, 1949
by LOWE & BRYDONE, PRINTERS, LTD., LONDON
from corrected sheets of the first edition.

PREFACE

For some five years, with collaborators who have supplied unfailing stimulus, I have sought to assess the range and power of a new approach to physical and engineering calculations. Devised originally for the computation of stresses in braced frameworks, on closer examination the notion of 'systematic relaxation of constraints' was seen to rest on a fundamental principle in Statics, and as such (by mathematical analogy) to have applications in far wider fields. Since the summer of 1935 we have extended it to problems of increasing difficulty, and as yet no insurmountable restriction has appeared.

We have not come to the end of our inquiry, but it has reached a stage at which review is possible, and several reasons have impelled me to attempt the task. First, the original papers are not commonly accessible to engineers, and of some our stocks of off-prints are exhausted,—I had wrongly estimated the probable demand. Secondly, as the scope of 'relaxation methods' has widened we have made, inevitably, some changes in notation and nomenclature. Thirdly, here as in most research the order of discovery has not been the order in which results are best presented. To these considerations lately another has been added which I find decisive: the trend of our most recent work has been towards problems in two dimensions, a field of less concern to engineering than to theoretical physics. We can deal now with most of the computational problems which confront an engineer, and a book can be attempted for his guidance which will have some claim to be termed complete.

Only a limited objective could have had, in war time, any chance of realization. I have sought to exhibit systematically one particular technique: I have not attempted a comprehensive treatise, and even basic matters like the Calculus of Finite Differences only receive such cursory notice as is necessary to the argument. Although in fact inevitable this restriction of aim may perhaps have advantages. Only time can decide the value of the new methods in relation to those used hitherto: what matter now are the range of problems to which they can provide solutions and the extent of the mathematical knowledge which they presume.

Regarding the power of relaxation methods, some indication is provided by the table of contents which follows; as to the knowledge

that they presume, the answer is that we have regarded no problem as solved until, for the actual computations, no more than the first four rules of arithmetic is required. What is of greater importance, even though a computer may not comprehend the theoretical basis of his calculations he will have, throughout, a mental picture of what he is doing; he will see his task as that of bringing unaccounted or 'residual' quantities within a specified margin of uncertainty. Whether regarded philosophically or practically, this is the essential feature by which the new mathematics differs from the old: it is 'mathematics with a fringe'. Regarded philosophically, thereby it takes account of the unavoidable uncertainty of physical data; and because the extent of that uncertainty is left, as it clearly should be, for decision in the light of practical experience, this feature should commend it to engineers.

In the initial preparation of material, and during its passage through the press, I have been given help which calls for grateful acknowledgement: first, by my secretary, Miss D. Castle, whose accurate typing has relieved me of much labour; secondly, by the unfailing readiness of the University Press to meet my wishes in regard to such matters as founts and the arrangement of the numerous tables; and finally, in the correction of the proofs, by many of those colleagues whose contributions are acknowledged in this book. For its shortcomings, and for any mistakes which have escaped detection, I am of course solely responsible. It has been brought to publication in a time of many distractions, and I can only hope that its errors are not numerous.

R. V. S.

OXFORD,
June, 1940

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I

RELAXATION METHODS APPLIED TO PIN-JOINTED FRAMEWORKS

Basic principles

1. RELAXATION methods as originally propounded were a means of calculating stresses in highly redundant pin-jointed frameworks† —a means which obviates the necessity of solving large numbers of simultaneous equations. As was to be expected they have since proved applicable to other problems, mathematically similar, which occur in theoretical and applied physics, and their essentials can be presented (Chap. V) in purely mathematical form: nevertheless it is convenient to approach that presentation by way of the problem for which they were devised, if only for the reason that thereby we obtain a convenient nomenclature.

Their basis as applied to frameworks was the notion that in any specified problem, although it may be difficult to obtain solutions by direct attack, it is easy to reverse this procedure and to calculate the forces which must be applied in order to maintain specified displacements. *We can always determine the error of a trial solution.* By itself this notion (which is the basis of all indirect methods) has little practical utility because a trial solution has little chance of being even approximately exact. But the Relaxation Method takes it farther by devising methods of systematic adjustment: we apply a series of **operations**, each one an indirect solution of particularly simple kind, and in this way we 'tune up' a trial solution (whether good or bad) until it conforms with some imposed standard of accuracy.

2. The simplest type of operation is the imposition of a **joint displacement** whereby one joint of the framework is displaced through a unit distance in some specified direction, all other joints being held fixed. It is clear that forces will be required for this purpose not only at the joint which is moved but also at every joint which is connected with it by some member of the framework, and further, that the forces will have directions which differ (in general) from

† A framework is termed highly redundant when many of its members could be removed without destroying its ability to sustain applied forces. The notion of 'pin joints' is an abstraction introduced for practical convenience (cf. § 90).

that of the displacement. We can calculate the forces at any joint of any framework provided that we can do so for each end of a single component member having any specified direction: the latter, accordingly, we term the **unit problem**, since its solution provides information regarding any joint displacement which we may desire to impose, and because when we know the forces entailed by every joint displacement which is possible, then we can deal with any specified loads applied to any specified (pin-jointed) framework. This last part of the calculation is the province of the relaxation process now to be described. We follow in computation a physical procedure which could (at least in imagination) be applied to an actual structure.

3. The ordinary 'screw-jack' (e.g. for automobiles) is a means whereby a controlled displacement may be imposed at some desired point. It is easy to imagine devices whereby the displacement may be recorded together with the load sustained at any instant, also to visualize an arrangement in which every joint of the structure which would normally be free to move is constrained by jacks of this kind (hereafter termed **constraints**) arranged so that they control its displacement in each, severally, of three perpendicular directions (or two, in the case of a 'plane framework'). Suppose that initially the joints are fixed in positions such that every member is unstrained: then evidently, when the external loads are applied, these will be taken wholly by the constraints. Suppose that subsequently one constraint is relaxed so that one joint is permitted to travel slowly† through a specified distance in some specified direction: then force will be transferred from that constraint to adjacent constraints and to the framework, and strain-energy will be stored in the latter. If the initial force on the relaxed constraint had a component in the direction of the travel, that constraint will be relieved and strain-energy will be stored at the expense of the potential energy of the external forces. All joints but the one being fixed, a table of 'standard operations', calculated as described above, will tell us how much force is transferred as the result of a specified displacement; therefore we can so adjust the displacement that the constraint is relieved either entirely or to any desired extent.

Now let this constraint be fixed in its new position, and let some other constraint be relaxed: evidently we can arrange matters so

† So that equilibrium is maintained, and vibrations are not excited.

that the second constraint is relieved of load, and the process can be repeated indefinitely. Of course, each relaxation will affect not only the force on the constraint which is relaxed but also the forces on neighbouring constraints, and loads will thus reappear on constraints which previous operations had relieved; but if at every operation the largest residual force is 'liquidated' (i.e. transferred by an appropriate displacement to adjacent constraints), then in general only forces of smaller magnitudes will be left for subsequent liquidation, because the external forces (which must be self-equilibrating) will tend to meet and cancel. Intuitively it is obvious that if at every stage the jack most heavily loaded is 'eased away' until it records a zero load, then eventually the force on every jack will be negligible, the framework sustaining practically the whole of the applied load-system.

4. If loads were applied to a real structure in this way, attention would be focused on the forces which at any instant are carried by the jacks. *The Relaxation Method reproduces this practical feature in computation*: that is to say, account is kept not of the forces sustained by the framework but of the loads which are taken by the visualized 'constraints'. Initially these loads are simply the external forces; but as it undergoes displacement the framework comes to take its share, and the **residual forces** (i.e. the forces still acting on the constraints) on the whole tend steadily to decrease. We say that in this way the external forces are **liquidated**: our aim is to liquidate them either completely or to an extent which is deemed sufficient, having regard to the margin of uncertainty which must always exist in practice *because the external load-system cannot be specified exactly.*

Consider the residual force ($\mathbf{X}_A$, say) which acts in the direction Ox on the constraint imposed at any joint A. It is the resultant (i) of the external force (X_A, say) which is imposed in this direction at A, and (ii) of the force (X_A, say) which is imposed by the framework on the constraint. That is,

$$\mathbf{X}_A = \mathsf{X}_A + X_A, \tag{1}$$

where X_A is specified but X_A depends on the displacements both of A *and of the joints which are connected with it.* Whatever operation is imposed, the change in $\mathbf{X}_A$ will (since X_A is invariable) be the change in X_A: it can be calculated when we know the nature of

the operation and the elastic properties of every member which is joined with A.

The 'unit problem' for pin-jointed frameworks

5. Thus we come to the 'unit problem' (§ 2): *A member MF of a freely jointed framework connects two joints M, F which have any position relative to one another. F being held fixed, displacements* **u**, v, **w** *are imposed on M. What forces, in consequence, are imposed by the member on the joints F and M?*

The answer is simple, given an expression for the extension of FM which results from the displacements **u**, v, **w**. This is (*Elasticity* § 43)†

$$L\epsilon = \Delta x.\mathbf{u}+\Delta y.\mathrm{v}+\Delta z.\mathbf{w}, \tag{2}$$

where L denotes the original length MF,

ϵ ,, ,, total extension (i.e. the increase of this length due to strain), and

Δx, Δy, Δz stand for (x_M-x_F), (y_M-y_F), (z_M-z_F) respectively,

x_M, y_M, z_M and x_F, y_F, z_F being the initial coordinates (relative to any fixed system of rectangular axes) of M and F. We observe that by definition

$$L^2 = (\Delta x)^2+(\Delta y)^2+(\Delta z)^2. \tag{3}$$

Let Ω_{MF} be a constant defining the elastic properties of MF and such that

$$P_{MF} = \text{total tension in } MF = (L^2\Omega\epsilon)_{MF}; \tag{4}$$

also let T_{MF} be the 'tension coefficient' of MF, defined (*Elasticity* § 102) by the relation

$$P_{MF} = L_{MF}.T_{MF}. \tag{5}$$

Then, by (2), (4) and (5),

$$T_{MF} = (L\Omega\epsilon)_{MF} = \Omega_{MF}(\Delta x.\mathbf{u}+\Delta y.\mathrm{v}+\Delta z.\mathbf{w}). \tag{6}$$

(If MF had uniform cross-sectional area A and were made of material having Young's modulus E, the value of Ω_{MF} would be EA/L^3.)

Now the forces imposed on F by reason of the tension in MF are given by

$$T_{MF}\times(\Delta x, \Delta y, \Delta z),$$

† References in this form are to the author's *Introduction to the Theory of Elasticity* (1st edit. 1936: Oxford Univ. Press).

and the forces imposed on M are equal and opposite. Therefore we have finally, on substitution from (6),

$$\left.\begin{aligned} -X_M &= X_F = \Omega_{MF}(\Delta x^2.\mathbf{u}+\Delta x.\Delta y.\mathbf{v}+\Delta x.\Delta z.\mathbf{w}),\\ -Y_M &= Y_F = \Omega_{MF}(\Delta y.\Delta x.\mathbf{u}+\Delta y^2.\mathbf{v}+\Delta y.\Delta z.\mathbf{w}),\\ -Z_M &= Z_F = \Omega_{MF}(\Delta z.\Delta x.\mathbf{u}+\Delta z.\Delta y.\mathbf{v}+\Delta z^2.\mathbf{w}),\end{aligned}\right\} \tag{7}$$

X_M, Y_M, Z_M and X_F, Y_F, Z_F being the component forces exerted by the member MF on M and F respectively, in consequence of displacements $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ imposed on M while F is held fixed.

Influence coefficients

6. The same results can be expressed more concisely in the notation of 'influence coefficients'. Denoting by $\widehat{xx}_A$, $\widehat{xy}_A$, $\widehat{xz}_A$ the forces in the x-direction which come on A as the result of unit displacements in the directions of x, y and z respectively, we may write

$$X_F = \widehat{xx}_F.\mathbf{u}+\widehat{xy}_F.\mathbf{v}+\widehat{xz}_F.\mathbf{w}, \quad ..., \text{etc.,} \tag{8}$$

and then from a comparison of (7) and (8) we have

$$\left.\begin{aligned} -\widehat{xx}_M &= \widehat{xx}_F = \Omega_{MF}(\Delta x)^2,\\ -\widehat{yy}_M &= \widehat{yy}_F = \Omega_{MF}(\Delta y)^2,\\ -\widehat{zz}_M &= \widehat{zz}_F = \Omega_{MF}(\Delta z)^2,\\ -\widehat{xy}_M &= \widehat{xy}_F = \Omega_{MF}(\Delta x.\Delta y) = \widehat{yx}_F = -\widehat{yx}_M,\\ -\widehat{yz}_M &= \widehat{yz}_F = \Omega_{MF}(\Delta y.\Delta z) = \widehat{zy}_F = -\widehat{zy}_M,\\ -\widehat{zx}_M &= \widehat{zx}_F = \Omega_{MF}(\Delta z.\Delta x) = \widehat{xz}_F = -\widehat{xz}_M.\end{aligned}\right\} \tag{9}$$

The quantities represented by $\widehat{xx}$,..., etc., are **influence coefficients.** The identities

$$\widehat{xy} = \widehat{yx}, \quad ..., \text{etc.,} \tag{10}$$

are examples of Maxwell's Reciprocal Relation (*Elasticity* § 10).

The fact that Δx, Δy, Δz appear in pairs, as products, in the expressions for the influence coefficients means that it is immaterial whether (as in § 5) we define them as (x_M-x_F), (y_M-y_F), (z_M-z_F) or in the opposite manner as (x_F-x_M), (y_F-y_M), (z_F-z_M), *provided that our definition is consistent as regards all three.* This is a convenient circumstance, since it eliminates the necessity of memorizing a particular convention.

7. Influence coefficients of the kind considered here are evidently properties of the member whose joints are moved. Given the dimensions and material of a framework, we may tabulate their values for

every member in the manner of Table I, which is self-explanatory.† *Since they are 'dimensional', definite units must be adopted.*

It should be remembered that while the elastic properties of a member cannot be specified with great precision, and accordingly the Ω's are only known to one or two significant figures, the values assigned to Δx, Δy, Δz, and to the quantities which are derived from them, must be consistent with the requirements of geometry. This consideration explains why an apparently unnecessary number of significant figures is retained in some items of Table I, notwithstanding that the Ω's are subject to errors of the order $\pm 0{\cdot}05$.

The operations table

8. Having a table of this kind we can revert to § 4 and calculate the forces (of type X_A) which the framework exerts upon the constraints in consequence of a displacement imposed on any joint. To fix ideas, let A be the joint which is moved, u_A the displacement permitted; and let B, C,..., K be the joints which in the framework are directly connected with A. We begin by considering the member AB.

In the operation considered, B is held fixed and A is moved. Hence, according to (7) and (8) with $\mathbf{u} = u_A$, $\mathbf{v} = \mathbf{w} = 0$,

$$\left.\begin{aligned} -X_A &= X_B = u_A\,\widehat{xx}_{AB}, \\ -Y_A &= Y_B = u_A\,\widehat{yx}_{AB}, \\ -Z_A &= Z_B = u_A\,\widehat{zx}_{AB}, \end{aligned}\right\} \tag{11}$$

whatever be the relative positions of A and B. Evidently we can deal in the same way with each of the other members involved; and so, whatever be the positions of B, C,..., K in relation to A, the displacement u_A (in a relaxation confined to this particular joint and direction) involves forces as under:

in the x-direction	*in the y-direction*	*in the z-direction*
$u_A \times$ xx_{AB} on B	$u_A \times$ $\widehat{yx}_{AB}$ on B	$u_A \times$ $\widehat{zx}_{AB}$ on B
xx_{AC} on C	$\widehat{yx}_{AC}$ on C	$\widehat{zx}_{AC}$ on C
		. . . .
xx_{AK} on K	$\widehat{yx}_{AK}$ on K	$\widehat{zx}_{AK}$ on K
$-u_A \sum_A [\widehat{xx}]$ on A;	$-u_A \sum_A [\widehat{yx}]$ on A;	$-u_A \sum_A [\widehat{zx}]$ on A.

† Some columns in Table I are blank for the reason that it relates to a *plane* framework (z = const.).

Table I

(Units: 1 ton weight; 1 foot.)

Column no.	1 Ω	2 Δx	3 Δy	4 Δz	5	6	7	8 $\widehat{xx}$	9 $\widehat{yy}$	10 $\widehat{zz}$	11 $\widehat{xy} = \widehat{yx}$	12 $\widehat{yz} = \widehat{zy}$	13 $\widehat{zx} = \widehat{xz}$
Method of derivation	Given	Given	Given	Given	1×2	1×3	1×4	2×5	3×6	4×7	3×5 or 2×6	4×6 or 3×7	2×7 or 4×5
Member													
AB	31·25	0	12	..	0	375	..	0	4,500	..	0	..	..
BC	8·45	20	0	..	169	0	..	3,380	0	..	0	..	..
CD	5·05	0	20	..	0	101	..	0	2,020	..	0	..	..
DA	6·75	20	−8	..	135	−54	..	2,700	432	..	−1,080	..	..
AC	3·20	20	12	..	64	38·4	..	1,280	460·8	..	768	..	..
BD	1·80	20	−20	..	36	−36	..	720	720	..	−720	..	..

The symbol $\sum_A$ is here used to denote a summation extending to every member of which one end is situated at A.

The effects of displacements v_A, w_A may be calculated similarly, v_A being associated with $\widehat{xy}, \widehat{yy}, \widehat{zy}$ and w_A with $\widehat{xz}, \widehat{yz}, \widehat{zz}$, as regards forces in the directions x, y, z respectively; and we can deal in the same way with any other joint displacement. Having completed the calculations we can present their results in an **operations table**, as exemplified in lines 1 (a) to 5 (b) of Table II.†

9. Each of these lines summarizes a series of calculations of the type described in § 8. As in Table I, a definite system of units is contemplated.

The numbering of the lines and columns should be noticed, since its scheme can be standardized with advantage. Column 2, for example, relates to X_B, and the displacement 'corresponding with' X_B (in the sense of *Elasticity* § 28) is u_B: lines relating to this joint-displacement are distinguished by the same number 2. There are two lines associated with any one joint-displacement: the first line of each pair, printed in italic figures, gives the forces associated with a unit displacement; in the second, the largest of these forces is reduced to 1,000 and the other forces (and the displacement) in proportion. The second line gives what is wanted for the calculations of the next stage, in which we seek to 'liquidate the residual forces' in the manner of § 4: we can quickly multiply or divide its terms by an integral number chosen so as to cancel the greater part of the largest force remaining on the constraints.

In the nature of the case the sum of the X's for all joints must vanish, also the sum of the Y's: thus a useful check can be imposed upon the detailed accuracy of the calculations, provided that all X's and Y's are included in each line of the table. Actually Table II relates to a problem in which displacements of the types denoted by u_C, u_D, v_D are prohibited by the stated conditions—i.e. by actual instead of merely visualized constraints (§ 3): consequently there are no lines relating to these joint-displacements, and therefore, while columns are included for the corresponding forces X_C, X_D, Y_D, they are not numbered and they are separated from the other columns by a double ruling. (Cf. footnote to Table II.)

10. The essentials of these calculations will be grasped most easily if they are applied to a simple example. In the problem which

† Lines 6 and 7 are explained in § 14.

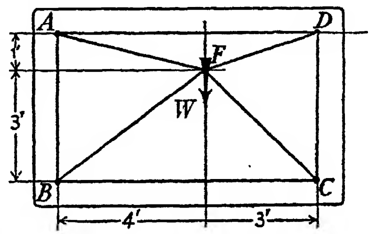
follows the absolute (as distinct from the relative) stiffnesses of the constituent members are not required for an answer to the question set; but to fix ideas it may be assumed in computation that

$$\left.\begin{aligned} \Omega &= 15{,}000/L^3 \text{ tons/(foot)}^3 \text{ units}\dagger \\ \text{and that} \qquad W &= 10 \text{ tons}, \end{aligned}\right\} \tag{12}$$

and subsequently the results can be generalized. Only one joint (F) can be moved, and that only in two directions (since the framework is plane): therefore only two 'operations' have to be considered.

Example

1. (*Oxford F.E.E.S.* 1938.) In the appended diagram, A, B, C, D are the corners of a rigid rectangular frame which is prevented from moving; FA, FB, FC, FD are uniform members, having the same cross-sectional area, which are pin-jointed to A, B, C, D and to one another at F. What forces are imposed on the frame at A, B, C and D when a load W is suspended from F?



The computation of influence coefficients is left to the reader, also the calculations (on the lines of § 8) which yield the following table of operations:

	X_A	X_B	X_C	X_D	X_F
$u_F = 1$	3,424	1,920	1,768·5	4,275	−11,387·5
$v_F = 1$	856	−1,440	1,768·5	−1,425	240·5

	Y_A	Y_B	Y_C	Y_D	Y_F
$u_F = 1$	856	−1,440	1,768·5	−1,425	+240·5
$v_F = 1$	214	1,080	1,768·5	475	−3,537·5

The direction Ox is taken from left to right, with Oy vertically downwards. The units are the foot and ton weight.

An answer can now be given to the question as set:—The displacements must be such as will liquidate the load (W) which acts initially, in the y-direction, on the constraint at F, without imposing force in the x-direction: hence, according to the above table, we must have (when $W = 10$ tons)

$$\begin{aligned} -11{,}387\cdot_5 u_F + 240\cdot_5 v_F &= 0 \quad (\text{direction } Ox), \\ 10 + 240\cdot_5 u_F - 3{,}537\cdot_5 v_F &= 0 \quad (\quad ,, \qquad Oy). \end{aligned} \tag{13}$$

† Cf. equation (4), § 5.

TABLE

igh foo

Operation no.	Nature of operation	(1) X_A	(2) X_B	(3) Y_A	(4) Y_B	(5) Y_C	X_C	X_D	Y_D
1 (a)	$u_A = 1$	*−3,980*	*0*	*312*	*0*	*768*	*1,280*	*2,700*	*−1,080*
1 (b)	$= 0{\cdot}2513$	−1,000	0	78_5	0	193	322	678	-271_5
2 (a)	$u_B = 1$	*0*	*−4,100*	*0*	*720*	*0*	*3,380*	*720*	*−720*
2 (b)	$= 0{\cdot}2439$	0	−1,000	0	175	0	825	175	−175
3 (a)	$v_A = 1$	*312*	*0*	*−5,393*	*4,500*	*461*	*768*	*−1,080*	*432*
3 (b)	$= 0{\cdot}1854$	58	0	−1,000	834	85_5	142_5	-200_5	80_5
4 (a)	$v_B = 1$	*0*	*720*	*4,500*	*−5,220*	*0*	*0*	*−720*	*720*
4 (b)	$= 0{\cdot}1916$	0	138	862	−1,000	0	0	−138	138
5 (a)	$v_C = 1$	*768*	*0*	*460·8*	*0*	*−2,480·8*	*−768*	*0*	*2,020*
5 (b)	$= 0{\cdot}4031$	309_5	0	186	0	−1,000	-309_5	0	814
6 (a)	$v_A = v_B = 1$	*312*	*720*	*−893*	*−720*	*461*	*768*	*−1,800*	*1,152*
6 (b)	$= 0{\cdot}6200$	193	446	−554	−446	286	476	−1,116	714
7 (a)	$v_A = -v_B = 1$	*312*	*−720*	*−9,893*	*9,720*	*461*	*768*	*−360*	*−288*
7 (b)	$= 0{\cdot}0510$	16	−37	−504	496	23	39	−18	−15

† N.B.—Since the operations affect (e.g.) X_A and $\mathbf{X}_A$ equally (§ 4), columns in this table might have been headed with symbols of either type. Here columns (1)–(5) have been headed in accordance with Table III, where the columns relate definitely to residual forces on constraints.

Solving these simultaneous equations we find that (in feet)

$$u_F = 0{\cdot}598 \times 10^{-4}, \qquad v_F = 28{\cdot}32 \times 10^{-4}, \tag{14}$$

and hence (using the table again) we have the solution given below.

	X_A	X_B	X_C	X_D	X_F
Initial forces .	..	..	..	..	..
$u_F = 0{\cdot}598 \times 10^{-4}$	0·205	0·115	$0{\cdot}105_5$	$0{\cdot}255_5$	−0·681
$v_F = 28{\cdot}32 \times 10^{-4}$	2·425	−4·081	5·010	−4·035	0·681
Total . .	2·630	−3·966	$5{\cdot}115_5$	$-3{\cdot}779_5$	0
	Y_A	Y_B	Y_C	Y_D	Y_F
	..	..	..	..	10
	0·051	−0·086	0·105	−0·085	+0·015
	0·605	3·060	5·005	1·345	−10·015
	0·656	2·974	5·110	1·260	0

These results will hold provided that

$$\left.\begin{aligned} \Omega &= 15{,}000/L^3 \text{ tons/(foot)}^3 \text{ units}, \\ W &= 10 \text{ tons}. \end{aligned}\right\} \tag{12) bis (}$$

Reducing Ω in any ratio $1 : k$ we shall have the same resultant forces but the displacements will be increased in the ratio $k : 1$; and when W is altered in the ratio $\lambda : 1$ all displacements and forces must be multiplied by the same factor. Thus we can at once deduce the solution for a load of W tons acting on members whose cross-section has any specified value.

The relaxation process

11. Reverting to our general description (§ 9), we now proceed to explain the final calculations whereby initial forces are 'liquidated' (§ 4). For this explanation the problem just considered will afford convenient illustrations.

Having derived equations (13), in § 10 we solved them as simultaneous equations. This was an easy matter as regards two equations, but much greater labour would have been entailed (and the same accuracy would have been more difficult to attain) if the equations had been more numerous. *The relaxation procedure is a means whereby simultaneous equations may be solved, not exactly, but with steadily increasing approximation.*

Referring again to the first table in § 10, we see that a unit displacement v_F will liquidate a residual force $Y_F = 3{,}537{\cdot}_5$ but will thereby introduce a force $X_F = 240{\cdot}_5$, a unit displacement u_F will

liquidate a residual force $X_F = 11{,}387{\cdot}_5$ but will thereby introduce a force $Y_F = 240{\cdot}_5$. Let us then impose in turn a displacement given by

$$v_A = \frac{10}{3{,}537{\cdot}_5} = 28{\cdot}29 \times 10^{-4} \text{ foot}$$

and a displacement given by

$$u_A = \frac{28{\cdot}29 \times 0{\cdot}0240_5}{11{,}387{\cdot}_5} = 0{\cdot}597 \times 10^{-4} \text{ foot.}$$

Thereby, starting with initial forces $Y_F = 10$, $X_F = 0$, we shall be left after two operations with residual forces $Y_F = 0{\cdot}0143_5$, $X_F = 0$, as the following tabular calculation shows:

	Y_F	X_F
Initial forces Y_F, X_F . . .	10	0
Effect of $v_A = 28{\cdot}29 \times 10^{-4}$ . .	-10	0·680
Residual after first operation . .	0	0·680
Effect of $u_A = 0{\cdot}597 \times 10^{-4}$ . .	$0{\cdot}0143_5$	$-0{\cdot}680$
Residual after second operation . .	$0{\cdot}0143_5$	0

Evidently the two operations combined have reduced Y_F by $10 - 0{\cdot}0143_5 = 9{\cdot}9856_5$ *without introducing a force* X_F: therefore, if both displacements had been increased in the ratio

$$\frac{10}{9{\cdot}9856_5} = 1{\cdot}00145,$$

we should have satisfied all the conditions of our problem. Thus by a different route we arrive at the result (14) given previously.

12. The last step in this argument would not be possible if we had many forces to liquidate—unless by chance we found a means of reducing all in the correct proportions. But even as it stands the calculation (here presented in a simple 'Relaxation Table') has reduced the largest 'unliquidated' or 'residual force' from 10 to about 1/70 ton, and by continuing to impose displacements as above we could bring the residual forces, ultimately, below any finite limit which may be imposed (although we shall not, however long we continue the process, arrive at an 'exact' solution). This is the essential principle of the relaxation method, which we now apply to a framework having less restricted freedom.†

† It is not suggested that in this particular example relaxation methods provide the most rapid approach to a solution: their advantages increase with the number of the joints. But the five degrees of freedom which characterize this example are sufficient to illustrate the relaxation procedure without entailing tables of inconvenient size.

Example

2. (*Camb. M.S.T.* 1932.) The framework shown in Fig. 1 (*a*) has one redundant member. E has the value 13,500 tons per sq. in., and the external forces, also the dimensions of the members, are as shown. Determine the action in every member under the external loading shown.

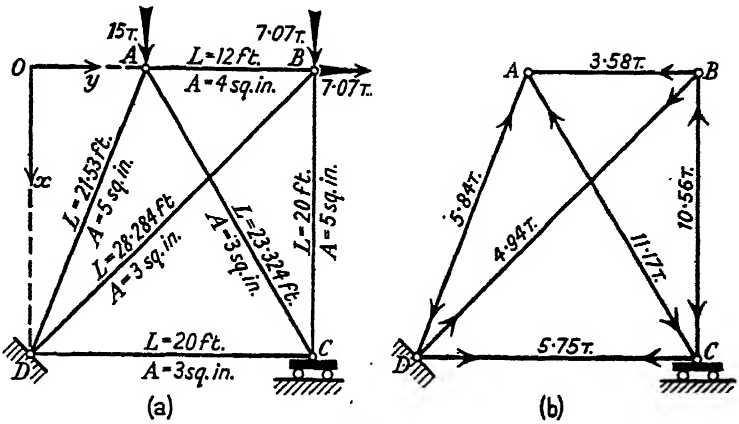


Fig. 1

Values of the influence coefficients, and the results of all permissible operations, have in fact been recorded for this example in Tables I and II. The joints free to move are A, B (in the direction both of x and y) and C (only in the direction of y). The initial forces acting on these joints and in these directions are recorded in the first line of the Relaxation Table III.

13. The procedure for liquidating the initial forces is a generalization of the procedure described in § 11; it is effected with the aid of the Operations Table II (§ 8). The largest of the initial forces is X_A (= 15 tons), and the corresponding operation (involving the displacement, u_A, which 'corresponds' with X_A: *Elasticity* §§ 28–30) is that numbered 1 in Table II. From that table we see that operation 1 (*b*) alters X_A by $-1{,}000$: therefore this operation with multiplier 15×10^{-3} will 'liquidate' X_A completely. We record the operation and multiplier in the second line of Table III, and an addition gives us (in line 3) new residual forces of which the largest is 7·07 tons.

In the next line, by applying Operation no. 2 (*b*) with multiplier 7×10^{-3}, we liquidate the greater part of X_B. Exact liquidation, which would entail a fractional multiplier, is not worth while because (as the result of subsequent operations) a residual force X_B will in

any event reappear. After this second operation the greatest of the residual forces has magnitude 8·29—somewhat greater than the previous largest figure.

14. The reader will now appreciate the significance of the operations numbered 6 and 7 in Table II. Examining Fig. 1 (*a*) he will realize that the joints *A* and *B* will in fact move roughly together, and through a distance great in comparison with what would be the displacement of either (under the same force) if the other were held fixed: for that reason it is convenient (as a means to the saving of labour) to have an operation in which v_A and v_B are equal, also (with a view to corrections required at a later stage) an operation in which v_A and v_B are equal and opposite. These operations are numbered 6 and 7 respectively in Table II: they are easily obtained (on the basis of the Principle of Superposition) by addition and subtraction of Operations 3 and 4 of the same table.

Evidently, as the second operation in § 13, no. 6 (*b*) would have been preferable to no. 2 (*b*). Applied now as a third operation with multiplier 10×10^{-3}, it renders Y_A and Y_B roughly equal and opposite, so that a subsequent application of Operation no. 7 (*b*), with multiplier -8×10^{-3}, liquidates both almost completely. The remaining operations in Table III call for no further description. At the finish we are left with residual forces in no case exceeding 0·02 ton—which may be taken as well within the margin of uncertainty regarding external loads.

15. When the approximation is deemed sufficient, the displacements may be calculated. The second column of Table II gives the displacements involved by 1,000 tons, and the first and second columns of Table III give the displacements imposed at different stages in the relaxation process. Adding the displacements involved by successive applications of the same operation, we find that

$$u_A \text{ (operation 1)} = 2{,}513 \times 10^{-7} \times (15+5-0{\cdot}5-0{\cdot}06) = 0{\cdot}004885 \text{ feet},$$

$$v_A \text{ (operations 6 and 7)} = 10^{-7} \times 6{,}200 \times (10+2{\cdot}5-0{\cdot}25-0{\cdot}04) + {} + 510(-8+0{\cdot}12) = 0{\cdot}007168 \text{ feet},$$

$$u_B \text{ (operation 2)} = 2{,}439 \times 10^{-7} \times (7+6-0{\cdot}2) = 0{\cdot}003122 \text{ feet},$$

$$v_B \text{ (operations 6 and 7)} = 10^{-7} \times 6{,}200 \times (10+2{\cdot}5-0{\cdot}25-0{\cdot}04) - {} - 510(-8+0{\cdot}12) = 0{\cdot}007972 \text{ feet},$$

$$v_C \text{ (operation 5)} = 4{,}031 \times 10^{-7} \times (7+0{\cdot}2-0{\cdot}12) = 0{\cdot}002854 \text{ feet}.$$

TABLE III

Operation	*Multiplier* × 10³	X_A	X_B	Y_A	Y_B	Y_C
	(Initial forces)→	15	7·07	..	7·07	..
1 (*b*)	15	−15	0	1·18	0	2·89
		0	7·07	1·18	7·07	2·89
2 (*b*)	7	0	−7·00	0	1·22	0
		0	0·07	1·18	8·29	2·89
6 (*b*)	10	1·93	4·46	−5·54	−4·46	2·86
		1·93	4·53	−4·36	3·83	5·75
7 (*b*)	−8	−0·13	0·30	4·03	−3·97	−0·18
		1·80	4·83	−0·33	−0·14	5·57
5 (*b*)	7	2·18	0	1·30	0	−7·00
		3·98	4·83	0·97	−0·14	−1·43
2 (*b*)	6	0	−6·00	0	1·05	0
		3·98	−1·17	0·97	+0·91	−1·43
1 (*b*)	5	−5·00	0	0·39	0	+0·97
		−1·02	−1·17	1·36	+0·91	−0·46
6 (*b*)	2·5	0·48	+1·12	−1·39	−1·11	0·72
		−0·54	−0·05	−0·03	−0·20	0·26
5 (*b*)	0·20	0·06	0	0·04	0	−0·20
		−0·48	−0·05	+0·01	−0·20	0·06
1 (*b*)	−0·5	0·50	0	−0·04	0	−0·10
		0·02	−0·05	−0·03	−0·20	−0·04
6 (*b*)	−0·25	−0·05	−0·11	+0·14	0·11	−0·07
		−0·03	−0·16	+0·11	−0·09	−0·11
2 (*b*)	−0·2	0	0·20	0	$-0·03_5$	0
		−0·03	0·04	0·11	$-0·12_5$	−0·11
7 (*b*)	0·2	0	−0·01	−0·10	0·10	0
		−0·03	0·03	0·01	$-0·02_5$	−0·11
5 (*b*)	−0·12	−0·04	0	−0·02	0	0·12
		−0·07	0·03	−0·01	$-0·02_5$	+0·01
1 (*b*)	−0·06	0·06	0	0	0	−0·02
		−0·01	0·03	−0·01	$-0·02_5$	−0·01
6 (*b*)	−0·04	0	−0·02	0·02	0·020	−0·01
		−0·01	0·01	0·01	0	−0·02

Thus the distortion of the framework is known, and the tensions in the members can be calculated by means of (2) and (4) with Ω's taken from Table I. In this way we arrive at the solution

Member . . .	*AB*	*BC*	*CD*	*DA*	*AC*	*BD*
Tension (*tons*) . .	3·54	−10·53	5·74	−5·83	−11·16	4·94

which may be compared with the 'correct' values given in Fig. 1 (*b*).

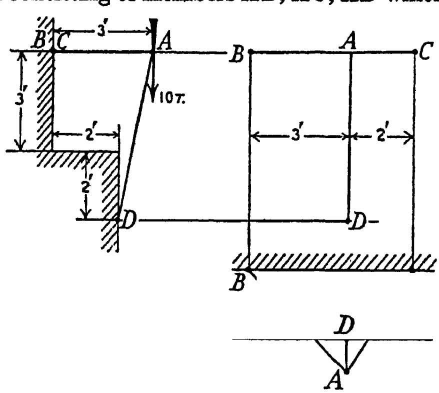
Since the final displacements are obtained by addition, errors may accumulate in the last decimal place. For this reason, if the framework under investigation is complex, it may be desirable to interrupt the standard procedure (Table III) when the approximation is getting near to what is deemed sufficient, calculate the total displacements up to this stage, and impose these as a starting assumption which can be improved by further relaxation if necessary. Similarly, if by arbitrary simplification of a problem we can deduce a solution believed to be approximate, this can be used as a starting assumption in the same way.

For reasons which have been explained in § 9, the forces X_C, X_D, Y_D have not to be liquidated, and on this account (to save space and labour) columns are not provided for them in the Relaxation Table. When the final displacements have been sufficiently determined, the last three columns of Table II may be used to calculate the corresponding forces of these types.

16. The examples which follow, like that of § 12, are not proposed as specially appropriate to the Relaxation Method; but their complexity is sufficient to reveal its essentials, and any one can be worked in a reasonable time.† The 'answers' appended are results obtained by D. J. Barclay and by R. W. G. Gandy; but any solution may be deemed exact for practical purposes in which the forces on constraints are liquidated to within 1 per cent. of their initial values.

Examples

3. (*Oxford F.E.E.S.* 1939.) A load of 10 tons is suspended from the joint A of a framework consisting of members AB, AC, AD which are freely jointed

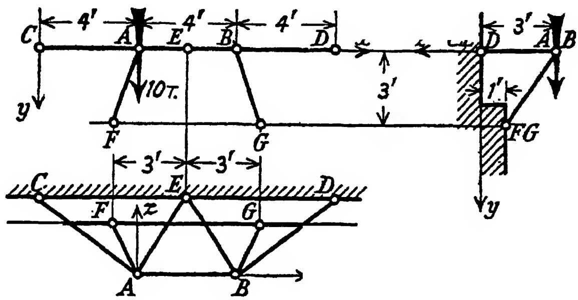


† Paper conveniently ruled for this purpose is sold under the description 'foreign money analysis paper'. It has been used from the first, at Oxford, for all Relaxation Tables containing many columns.

to one another and to points B, C, D of a rigid wall. Each member is of steel ($E = 13{,}500$ tons per sq. in.) and has a cross-section of 1 sq. in.

Calculate A's vertical displacement and the forces exerted on the wall at B, C and D. [*Ans.* $4{\cdot}02 \times 10^{-3}$ (feet).]

4. The pin-jointed framework shown is symmetrical with respect to a plane through E and perpendicular to the rigid wall to which it is affixed. All members are of steel ($E = 13{,}500$ tons per sq. in.) and have cross-sectional area 1 sq. in. A load of 10 tons is suspended from A as shown.



Calculate the displacements of A and B, and the forces applied to the wall at C, D, E, F, G. (*N.B.*—The problem may be simplified by dividing it into two parts and taking account of the symmetry. Thus 10 tons at A is equivalent to two load systems superposed, viz.—

(i) 5 tons at each of A and B;

(ii) 5 tons at A and -5 tons at B.)

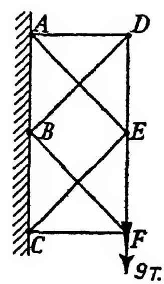
[*Ans.* $u_A = 949 \times 10^{-6}$, $v_A = 5{\cdot}95 \times 10^{-3}$, $w_A = -1{\cdot}98_5 \times 10^{-3}$ ft.
$u_B = 516 \times 10^{-6}$, $v_B = -213 \times 10^{-6}$, $w_B = 62{\cdot}_5 \times 10^{-6}$ ft.]

5. (*Oxford F.E.E.S.* 1938.) The pin-jointed steel frame shown is attached at A, B and C to a rigid vertical wall. All angles are either 45° or 90°. DE, EF are rigid, and the other members have cross-sections as under:

AE, EC, DB, BF, 1 sq. in.,
AD, CF, $\sqrt{2}$ sq. in.

Calculate the tensions in AD, CF when a load of 9 tons is suspended from F.

[*Ans.* 2 tons tension, 2 tons compression.]



RECAPITULATION

17. A reader who has followed this account of the Relaxation Method, and who by working one or more of the above examples has acquired some grasp of its practical details, will appreciate certain features which distinguish it from more orthodox methods of solution. In the first place attention is concentrated not on the

quantities whose values are sought (the stresses or joint displacements in the required configuration) but on the quantities which are specified (the external loads); the object being to 'liquidate' these (i.e. to determine their effects) not exactly, but *within some margin covered by the practical margin of uncertainty* (§ 4). This would appear to be a logical approach, since it is only in invented problems that the data can be specified with certainty: in practice loadings are uncertain to the same extent (at least) as the elastic properties of the structure.

A solution obtained by relaxation methods will hardly ever be exact in the mathematical sense; at every stage in the liquidation process some residual forces will remain—i.e. actions not accounted for. But if these (for example) have magnitudes in all cases less than 0·1 ton, and if the specified loads have a margin of uncertainty which exceeds this figure, then evidently further 'approximation' is not only unnecessary but meaningless. It is as likely to give a solution farther from, as a solution nearer to, the truth.

18. Secondly, although attention is thus concentrated on residual forces, the operations required for their liquidation are recorded, and so at the finish we have values for joint-displacements from which, by means of (2) and (4), the action in every member can be computed. Thus the Relaxation Method provides information regarding strains as well as stresses: by orthodox methods, e.g. those based on Castigliano's Principle of Minimum Strain-Energy, we determine stresses but cannot (without much additional labour) deduce the resulting distortion of the framework.

Thirdly, at every step in the relaxation process we start, in effect, afresh; preceding operations have given us displacements which constitute a new starting assumption. This means that we can at any time eliminate errors which may have entered either by mistakes in computation or by accumulation of errors in the last significant figure of our recorded quantities (§ 15); and equally, if by intuition or by assumptions not strictly warranted we can obtain some approximation to the required solution, full advantage of this circumstance can be taken by applying, as a first step, the displacements which appear in that approximation. The main features of the distortion will usually be intuitively evident to an experienced engineer: the Relaxation Method seems to be alone in permitting him to use his intuition without putting undue trust in its accuracy.

19. Fourthly, once the influence coefficients have been determined the calculations involve purely numerical quantities; there is no occasion to consider the directions of forces or members, and on that account 'space frameworks' (i.e. frameworks having extension in three dimensions) entail no greater difficulty than frameworks which are plane (i.e. have extension in only two). The actual computations (subsequent to those of Table II) could be performed by calculators having no knowledge of elastic theory; and the reader will have noticed that they involve no processes other than simple addition, subtraction, multiplication, and division. Frequent checks can be imposed (§ 9) of a kind which make it very improbable that chance errors will remain undetected; and in any event they will not have any disastrous result because (§ 18) in effect we start afresh at every step.

20. Finally (and it is perhaps the most significant feature of the Relaxation Method from a practical point of view), redundancy has almost no effect upon the time required for a solution. This will increase with the number of joints which can be moved; but the number of the members merely affects the time expended in preparing Tables I and II, and thereafter is quite immaterial.

We have seen on the other hand (§ 14) that the progress of approximation, though in general continuous, may be slow unless special devices are employed to accelerate it (e.g. the last two operations of Table II). Rapidity being essential in a method intended for practical use, this would be a serious drawback if not corrected: fortunately it can be remedied very simply, by the devices known as Block and Group Relaxations. These and other devices making for acceleration will be described in Chapter IV, after the basic principles stated here in relation to pin-jointed frameworks have been extended to cover frameworks which have rigid joints.

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II

RELAXATION METHODS APPLIED TO CONTINUOUS GIRDERS

21. We noticed in § 20, as an important feature of Relaxation Methods, that in them the labour of computation depends almost entirely on the number of joints in a framework, hardly at all upon the number of its members. Redundancy loses the importance which it has in more orthodox treatment, and on that account the new methods should have still greater advantages as applied to frameworks having rigid joints.

Consider, for example, the plane structure shown in Fig. 13 (p. 48). If all its members were hinged freely to one another and to the supporting wall, their actions (purely longitudinal) would be determinable from purely statical considerations; we should have four unknowns and four equations relating them—namely, two conditions of equilibrium at each of B and C. When on the other hand AB, BC and CD are rigidly connected as shown, there is a possibility of shear and bending moment in every one of these members, and on that account the number of unknowns is four, even after full use has been made of statical conditions.

No change in principle is entailed when Relaxation Methods are applied to structures of this kind, but actions will now be exerted not only when a joint is moved but also when it is rotated, and to define a configuration completely we must specify not only the positions but the directions of the ends of each constituent member. At points like A and D in Fig. 13, where members are built into rigid supports, neither displacement nor rotation can occur; but at other joints we have to contemplate (in the manner of § 3) not only 'jacks' which control displacement but also constraints of a kind which can control rotation. Constraints of the first type will sustain 'residual forces', those of the second type will sustain 'residual moments'.

22. As an approach to more difficult examples we now consider the problem of a straight girder resting on three or more supports and loaded transversely (Fig. 2a). In general, bending moments will be operative at every support, but only those at the outermost supports can be controlled and therefore specified: the problem will be solved when the others have been determined, because then the

bending moment at any intermediate section can be inferred. Orthodox methods make use of Clapeyron's 'Theorem of Three Moments', which supplies a relation between the bending moments at any three consecutive supports;† but the form of this relation is not very convenient, and it is easy to make mistakes in formulating the simultaneous equations.

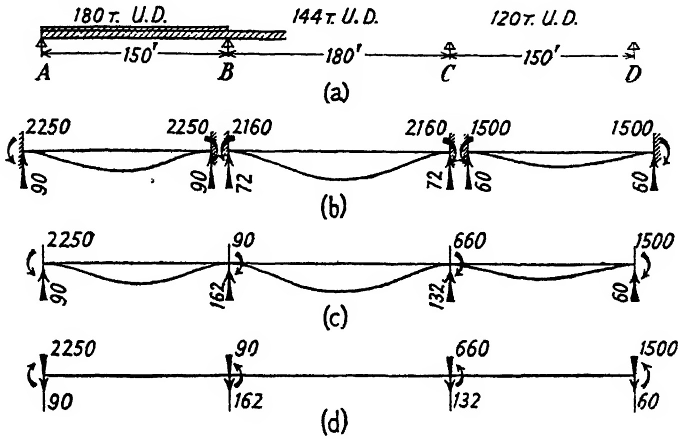

FIG. 2

To bring this problem within range of Relaxation Methods we must first reduce it to one in which applied forces and moments act only at the supported sections: that is to say, we must dispose of the transverse loadings on the several spans. We have recourse for this purpose to the Principle of Superposition,‡ dividing the problem into two parts in a manner which will be clear from Figs. 2. Treating each span severally as a beam clamped at both ends, we determine by standard methods (i) its deflected form and (ii) the clamping couples and supporting forces (Fig. 2*b*) which the transverse loading calls into play: then, assuming those actions to be applied as concentrated moments and forces, we have a form of deflexion for the complete girder (Fig. 2*c*) which involves neither deflexion nor slope at any of the supported sections, and so does not entail any action on the constraints. But actually the assumed moments and forces will *not* be operative: therefore we must calculate, and superpose on those of Fig. 2*c*, the deflexions entailed by a second system of concentrated

† Cf. *Elasticity* § 70. The number of such relations which can be formulated is equal to the 'order of redundancy', i.e. two less than the number of supports.

‡ *Elasticity* §§ 4–6.

moments and couples (Fig. 2 *d*), equal in magnitude but opposite in sign to the first.

23. Thus modified the problem is tractable by Relaxation Methods, because now the external forces and moments (including the reactions of the supports) act only at a few sections. We can imagine two adjustable constraints as operating at each supported section, one controlling the vertical displacement, the other the slope of the girder at that section. Initially the applied forces and moments (Fig. 2 *d*) are taken wholly on the constraints: subsequently, by systematic relaxation in the manner of § 13, we transfer them to the girder and its supports. The process can be stopped when, in the judgement of the designer, only negligible actions are left on the constraints.

A constraint for the prevention of vertical displacement will not be required at a section where the support is specified as rigid; for there displacement will be prevented, so the only quantity to be determined is the slope. But in practice it is sometimes necessary to make allowance for elasticity of the supports, and in that event vertical constraints will be required. (The Relaxation Method is specially useful in such cases, because elastic supports make the problem much more intricate when treated by orthodox methods.)

The 'unit problem' for continuous girders

24. Just as, for pin-jointed frameworks, everything reduced to the unit problem discussed in § 5 of Chapter I, so, in this instance, all that we require for the preparation of a table of standard operations is a solution of the 'unit problem' shown in Fig. 3 *a*. Of the straight member *mf*, by the agency of constraints, one end *f* is held fixed both in position and in direction, the other end *m* is deflected transversely through a distance δ and rotated through an angle r. We require expressions in terms of δ and r for the forces and moments which the member, in consequence, imposes on the constraints.

We shall treat this unit problem by orthodox methods (*Elasticity* §§ 192–3), assuming that the member, of length L, has uniform flexural rigidity B. In Fig. 3 *a*, M_0 and V_0 are the moment and reaction *applied to the member* at the moved end *m*, which we take as origin of x. Then

$$-By'' = \text{sagging b.m. at a distance } x \text{ from the origin}$$

$$= M_0 + V_0 x.$$

Integrating with allowance for the known conditions at $x = 0$, we have successively

$$-By' = M_0 x + \tfrac{1}{2}V_0 x^2 - Br$$

and
$$-By = \tfrac{1}{2}M_0 x^2 + \tfrac{1}{6}V_0 x^3 - Brx - B\delta.$$

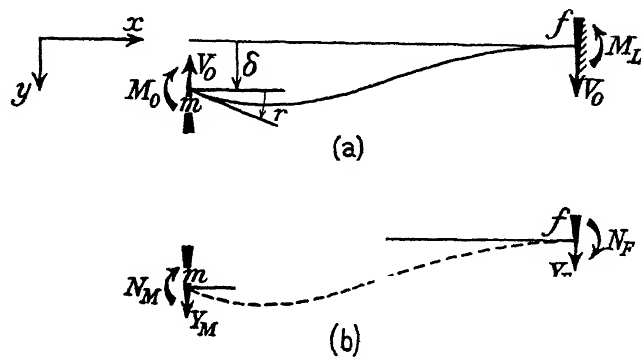

FIG. 3

Hence, since $y = y' = 0$ when $x = L$, we have the relations

$$M_0 L + \tfrac{1}{2}V_0 L^2 = Br,$$

$$\tfrac{1}{2}M_0 L^2 + \tfrac{1}{6}V_0 L^3 = B(rL + \delta),$$

and it follows that

$$\left.\begin{aligned} V_0 &= -6\frac{B}{L^3}(2\delta + Lr), \\ M_0 &= 2\frac{B}{L^2}(3\delta + 2Lr). \end{aligned}\right\} \qquad \text{(i)}$$

At the fixed end f ($x = L$), the force applied to the member must be V_0 in the direction of y, and the bending moment (sagging) is given by

$$M_L = M_0 + V_0 L = -2\frac{B}{L^2}(3\delta + Lr). \qquad \text{(ii)}$$

We can now write down expressions for the actions applied *to the constraints* (Fig. 3 *b*). Evidently the actions V_0, M_0, M_L on the girder will entail equal and opposite actions on the constraints: for the latter we must adopt a uniform sign-convention, and we shall say that forces (Y) are positive when directed along Oy, moments (N)

positive when clockwise like r. On this understanding

$$
\begin{aligned}
-Y_M = Y_F = -V_0 &= 6\frac{B}{L^3}(2\delta+Lr),\\
N_M = -M_0 &= -2\frac{B}{L^2}(3\delta+2Lr), \qquad \text{(iii)}\\
N_F = M_L &= -2\frac{B}{L^2}(3\delta+Lr),
\end{aligned}
$$

when (as in the diagrams) m is to the left of f.

The other case (m to the right of f) is obtainable by a simple 'reflection' of the diagrams. The downward displacement is δ as before, but the clockwise rotation is now $-r$; the expressions for Y_M, Y_F are unchanged, those for N_M, N_F (clockwise couples being regarded as positive) have their signs reversed.

Influence coefficients

25. The results for both cases can be combined in common formulae if we introduce the cosine l of the angle made with Ox by a line drawn from m to f. (Thus $l = 1$ when m is to the left of f, $l = -1$ when m is to the right.) In the notation of 'influence coefficients' (§ 6) we write

$$
\left.\begin{aligned}
Y &= \widehat{yy}\,.\mathbf{v}+\widehat{yr}\,.\mathbf{r},\\
N &= \widehat{ry}\,.\mathbf{v}+\widehat{rr}\,.\mathbf{r},
\end{aligned}\right\} \qquad (1)
$$

$\mathbf{v}$ denoting the imposed displacement in the direction of y, and $\mathbf{r}$ the imposed clockwise rotation. Then we have

for the fixed end:

$$
\left.\begin{aligned}
\widehat{yy}_F &= 12\frac{B}{L^3}, \qquad \widehat{rr}_F = -2\frac{B}{L},\\
-\widehat{ry}_F &= \widehat{yr}_F = 6\frac{B}{L^2}l;
\end{aligned}\right\} \qquad (2)
$$

for the moving end:

$$
\left.\begin{aligned}
\widehat{yy}_M &= -12\frac{B}{L^3}, \qquad \widehat{rr}_M = -4\frac{B}{L},\\
\widehat{ry}_M &= \widehat{yr}_M = -6\frac{B}{L^2}l.
\end{aligned}\right\} \qquad (3)
$$

It should be observed that the forces, but not the moments, have equal and opposite values at the two ends. The moments in fact have the same sign, as we might have foreseen by considering the curvatures (Fig. 3 a).

Having the formulae (1)–(3) we can at once calculate the forces and moments which will come upon constraints by reason of displacement or rotation imposed at any supported section of a continuous girder. Thus (Figs. 2), if constraints prevent A and C from moving while a displacement or rotation is imposed at B, then on account of the stiffness of AB and BC forces Y_A, Y_B, Y_C and moments M_A, M_B, M_C will be brought into play. The values of

Y_A, M_A will be given by (1) and (2) with $l = -1$,
Y_C, M_C ,, ,, ,, ,, $l = +1$,
Y_B, M_B will consist of parts contributed by AB and given by (1) and (3) with $l = -1$, also of parts contributed by BC and given by (1) and (3) with $l = 1$.

In general, if a displacement v_B is permitted it will entail an addition to Y_B (*of negative sign*) in consequence of the elastic resistance of the support. Allowance for this must be made in the table of operations.

Examples illustrative of the relaxation procedure

26. Understanding of the relaxation procedure will be gained most easily by working particular examples. We shall take first a case in which the supports are rigid: Figs. 2 relate to the example which follows.

Example

1. (*Camb. M.S.T.* 1910.) A railway line for a single railway is constructed with two main girders continuous over four supports, forming two side spans each 150 ft. long and a centre span 180 ft. long. The four supports are at the same level, the girders are of uniform section, and the ends are free from bending moment. The dead load on each main girder is 0·8 ton per ft. run, and the live load 0·4 ton per ft. run. Taking the case when a side span alone carries live load, determine the bending moments over the supports. Sketch the B.M. diagram, and determine the pressures on the four supports.

First we have to determine (cf. § 22) the clamping couples and supporting forces for each span considered severally: the necessary data are provided in Fig. 2*a*, this problem being No. 6 of the 'Standard Cases of Deflexion in Girders' (*Elasticity*, p. 191). Taking the foot and the ton weight as units we have

for the left-hand span: $L = 150$, $W = 180$, $M_0 = 15\times 150 = 2{,}250$,
,, central ,, $L = 180$, $W = 144$, $M_0 = 15\times 144 = 2{,}160$,
,, right-hand ,, $L = 150$, $W = 120$, $M_0 = 1{,}500$,

as shown in Fig. 2*b*. The terminal actions together with the trans-

verse loadings give a deflected form of the type of Fig. 2*c*: because they are not in fact applied, we must investigate (and superpose when found) the deflexions which result from the equilibrating system shown in Fig. 2*d*.

27. Since the supports are rigid, the only influence coefficients which enter into our calculations are $\widehat{yr}$ and $\widehat{rr}$. It is clear that the pressures on supports will not depend upon the absolute magnitude of the flexural rigidity,† and accordingly we may give B any convenient value. We shall take B as having the value 135,000 (for all spans):‡ then according to (2) and (3), for the outer spans

$$\left.\begin{array}{ll} \widehat{yr}_F = 36l, & \widehat{rr}_F = -1{,}800, \\ \widehat{yr}_M = -36l, & \widehat{rr}_M = -3{,}600, \end{array}\right\} \quad \text{(i)}$$

and for the central span

$$\left.\begin{array}{ll} \widehat{yr}_F = 25l, & \widehat{rr}_F = -1{,}500, \\ \widehat{yr}_M = -25l, & \widehat{rr}_M = -3{,}000, \end{array}\right\} \quad \text{(ii)}$$

when the units are 1 foot, 1 radian, 1 ton weight.

To find the consequences of a unit (clockwise) rotation of the constraint at A (Fig. 2*a*) we must (§ 25) put $l = 1$ in (i). Then we have

$$\left.\begin{array}{l} r_A = 1.\ Y_A = -36,\ Y_B = 36,\ N_A = -3{,}600,\ N_B = -1{,}800, \\ \text{and so, proportionately, for a rotation } 1/36, \\ r_A = \tfrac{1}{36}\colon Y_A = -1,\quad Y_B = 1,\quad N_A = -100,\quad N_B = -50. \end{array}\right\} \quad \text{(iii)}$$

To find the consequences of a unit rotation of the constraint at B we must (§ 25) put $l = -1$ in (i), $l = +1$ in (ii). Then we have

$$\left.\begin{array}{lll} r_B = 1\colon & & \\ Y_A = -36, & Y_B = 36-25 = 11, & Y_C = 25, \\ N_A = -1{,}800, & N_B = -3{,}600-3{,}000 = -6{,}600, & N_C = -1{,}500, \end{array}\right\} \quad \text{(iv)}$$

and these figures can be multiplied as before. Similarly, for a unit rotation of the constraint at C we have

$$\left.\begin{array}{lll} r_C = 1\colon & & \\ Y_B = -25, & Y_C = 25-36 = -11, & Y_D = 36, \\ N_B = -1{,}500, & N_C = -3{,}000-3{,}600 = -6{,}600, & N_D = -1{,}800, \end{array}\right\} \quad \text{(v)}$$

and for a unit rotation of the constraint at D

$$r_D = 1\colon \quad Y_C = -36,\ Y_D = 36,\ N_C = -1{,}800,\ N_D = -3{,}600. \quad \text{(vi)}$$

† The deflexions will be inversely proportional to B.

‡ This figure (chosen simply for convenience) implies a steel girder having $I = 1{,}440$ in.^{4} units approximately. Actually (in order to keep the stresses within reasonable limits) it would need to be increased in some ratio of the order of 250 : 1 (cf. § 30).

For convenience these results should be embodied in a 'Table of Standard Operations' as shown below. Then we are in a position to liquidate by a relaxation process the applied moments given in Fig. 2*d*.

TABLE IV. *Operations Table for Continuous Girder*

(Units: 1 foot; 1 ton weight; 1 radian. Cf. § 9 and Table II.)

No. and nature of operation	N_A	N_B	N_C	N_D	Y_A	Y_B	Y_C	Y_D
1 (*a*) $r_A = 1$	*−3,600*	*−1,800*	..	..	*−36*	*36*	..	..
1 (*b*) $r_A = \frac{1}{36}$	− 100	− 50	..	..	− 1	1	..	..
2 (*a*) $r_B = 1$	*− 1,800*	*−6,600*	*−1,500*	..	*−36*	*11*	*25*	..
2 (*b*) $r_B = \frac{1}{66}$	− 27·3	− 100	− 22·75	..	− 0·546	0·167	0·379	..
3 (*a*) $r_C = 1$	..	*−1,500*	*−6,600*	*−1,800*	..	*−25*	*−11*	*36*
3 (*b*) $r_C = \frac{1}{66}$	..	− 22·75	− 100	− 27·3	..	− 0·379	− 0·167	0·546
4 (*a*) $r_D = 1$	..	..	*−1,800*	*−3,600*	..	..	*−36*	*36*
4 (*b*) $r_D = \frac{1}{36}$	..	..	− 50	− 100	..	..	− 1	1

28. The relaxation process (concerned in this problem only with the liquidation of moments) is not reproduced here in full, but is summarized in Table V.† (It is effected in a relaxation table of exactly similar form to that of Table III, Chap. I.) It leaves only negligible moments on the constraints—namely, those shown in the last line of Table V. The *forces* (which come not on the constraints but on the fixed supports) are given in the last four columns.

The slopes at A, B, C, D which result from the operations recorded in the first column are (for the assumed value $B = 135{,}000$)

$$r_A = \frac{26\cdot70}{36} = 0\cdot742, \qquad r_B = -\frac{15\cdot41}{66} = -0\cdot2334,$$

$$r_C = \frac{5\cdot11}{66} = 0\cdot0774, \qquad \frac{16\cdot39}{36} = -0\cdot456 \text{ (radians)}. \quad (4)$$

TABLE V. *Summary of Relaxation Table for Continuous Girder*

Operation and multiplier	N_A	N_B	N_C	N_D	Y_A	Y_B	Y_C	Y_D
1 (*b*) × 26·70 . .	−2,670	−1,335	..	..	−26·70	26·70	..	..
2 (*b*) ×−15·41 . .	421	1,541	350_5	..	8·41	−2·57	−5·84	..
3 (*b*) × 5·11 . .	..	− 116_{25}	−511	− 139_5	..	$-1\cdot93_5$	$-0\cdot85_5$	2·79
4 (*b*) ×−16·39 . .	..	..	819_5	1,639	..	..	16·39	−16·39
(*a*) Totals . . .	−2,249	89·75	659	1,499·5	−18·29	$22\cdot19_5$	$9\cdot69_5$	−13·60
(*b*) Initial actions. .	2,250	−90	−660	−1,500	90	162	132	60
(*c*) Left on constraints .	+1	−0·25	−1	−0·5	71·71	$184\cdot19_5$	$141\cdot69_5$	46·40

† The order of the operations is arbitrary, and the final results of the relaxation process (if correctly performed) will be independent of this order. Therefore the summary gives all that is required. Bold letters distinguish actions on *constraints*.

Neither moments on constraints nor forces on supports will be imposed by superposition of the deflexions shown in Fig. 2 *c*, since these involve no change of slope at any support and since the actions are in equilibrium both statically and elastically. Therefore our final values for the pressures on the four supports are (line (*c*) of Table V)

$$Y_A = 71{\cdot}7, \quad Y_B = 184{\cdot}2, \quad Y_C = 141{\cdot}7, \quad Y_D = 46{\cdot}4 \text{ tons.} \tag{5}$$

This is part of the required solution.

29. We now deduce the bending moments in the girder (taken as positive when 'sagging') from a knowledge of the moments and reactions which come upon it at the supports. This is a purely statical problem: If M_0 is the bending moment at the left-hand end of any span, and if F_0 is the (upward) shear force at that end, then at the other end the bending moment will be $M_0+F_0\,L$, L being the length of the span in question. F_0 is the sum of the upward forces to left of the section considered.

Line (*a*) of Table V, since it gives the *clockwise* couples and *downward* forces which are transferred *through* the girder to the constraints, also gives the *counterclockwise* couples and *upward* forces which come *upon* the girder. Proceeding as above we find that the bending moments which result from the relaxation process are:

$$\left.\begin{array}{l}
\text{immediately to the right of } A,\\
\qquad 2{,}249,\\
\text{immediately to the left of } B,\\
\qquad 2{,}249+150\times(-18{\cdot}29) = -494{\cdot}5,\\
\text{immediately to the right of } B,\\
\qquad -494{\cdot}5-89{\cdot}75 = -584{\cdot}25,\\
\text{immediately to the left of } C,\\
\qquad -584{\cdot}25+180\times(-18{\cdot}29+22{\cdot}19_5 = 3{\cdot}90_5) = 118{\cdot}65,\\
\text{immediately to the right of } C,\\
\qquad 118{\cdot}65-659 = -540{\cdot}35,\\
\text{immediately to the left of } D,\\
\qquad -540{\cdot}35+150\times(3{\cdot}90_5+9{\cdot}69_5 = 13{\cdot}6) = 1{,}499{\cdot}6_5,\\
\text{immediately to the right of } D,\\
\qquad 1{,}499{\cdot}6_5-1{,}499{\cdot}5 = 0{\cdot}1_5 \text{ (negligible).}
\end{array}\right\} \tag{6}$$

The corresponding values of the bending moments in Fig. 2 *b* are

$$-2{,}250, \quad -2{,}250, \quad -2{,}160, \quad -2{,}160, \quad -1{,}500, \quad -1{,}500, \quad 0.$$

Adding these and (6), we find (as we should expect) that within a

negligible margin of 1 ton-foot the discontinuities at the supports are removed and the terminal moments are zero. We have, finally, as estimates for the bending moments at the supports:

$$M_A = 0, \quad M_B = -2{,}744{\cdot}5, \quad M_C = -2{,}041, \quad M_D = 0. \qquad (7)$$

This completes the solution. (*N.B.*—Since 'sagging' bending moments have been taken as positive, actually the bending moments at B and C are 'hogging'.)

30. Excepting the tabular presentation, everything in our working of this problem is identical with what would have been entailed by an application of the 'Moment Distribution Method' of Professor Hardy Cross (Ref. 2). Continuous beams present the one problem as applied to which the two methods are exactly equivalent (cf. § 89).

As was forecast in § 27, both (5) and (6) are unaffected by the numerical value assumed for B. This is evident if we reflect that the initial actions given in line (*b*) of the Relaxation Table V are independent of this value, while the consequences of unit rotations (in the Operations Table IV) are all proportional to B: therefore the rotations required for liquidation will be inversely proportional to B, and this is the reason why the values given in (4) are so large. Our assumed value $B = 135{,}000$ should be multiplied by about 250 in order that the stresses may be reasonable (I will then be of the order of $17{\cdot}5$ foot^4 units), and with this value the rotations in (4) will be correspondingly reduced.†

31. The absolute value of B is important when (as in the example which follows) the supports are elastic. For then the pressures are distributed in a manner which depends upon the relative elasticity of the girder and of the supports; so that if for convenience we decrease B in numerical work, then we must decrease in proportion the force required to depress a support through any specified distance.

Example

2. Work Example 1 on the assumptions (i) that the supports at A, C and D (Fig. 2) are rigid but that the support at B yields elastically by 1 inch for every 100 tons which it sustains, and (ii) that the flexural rigidity is 33,750,000 tons foot^2 units.‡

To allow for the new factor of a yielding support (i) an additional constraint must be imagined to operate at B and (ii) we must investigate the consequences of a unit vertical displacement (i.e. 1 foot)

† The maximum slope (at A) will be of the order 0·2 degree.

‡ i.e. $135{,}000 \times 250$ (cf. § 30).

imposed at B. The influence coefficients $\widehat{yy}$ and $\widehat{ry}$ will enter into this calculation, both for AB with $l = -1$ and for BC with $l = +1$. Inserting the new value for the flexural rigidity, we have from (2) and (3)

$$\left.\begin{aligned}
&\text{for } AB\text{:}\\
&\qquad -\widehat{yy}_M = \widehat{yy}_F = 12 \times \frac{3{,}375\times 10^4}{150^3} = 120,\\
&\qquad \widehat{ry}_M = \widehat{ry}_F = 6 \times \frac{3{,}375\times 10^4}{150^2} = 9{,}000,\\
&\text{for } BC\text{:}\\
&\qquad -\widehat{yy}_M = \widehat{yy}_F = 12 \times \frac{3{,}375\times 10^4}{180^3} = 69{\cdot}5,\\
&\qquad \widehat{ry}_M = \widehat{ry}_F = -6 \times \frac{3{,}375\times 10^4}{180^2} = -6{,}250.
\end{aligned}\right\} \quad \text{(i)}$$

Therefore, *as the result of flexure in the girder*, a unit displacement v_B imposes forces and moments as under:

$$\left.\begin{aligned}
&Y_A = 120, && Y_B = -120-69{\cdot}5 = -189{\cdot}5, && Y_C = 69{\cdot}5,\\
&N_A = 9{,}000, && N_B = 9{,}000-6{,}250 = 2{,}750, && N_C = -6{,}250.
\end{aligned}\right\} \quad \text{(ii)}$$

On the constraint at B, *in consequence of its elastic resistance*, a unit displacement v_B (i.e. 1 foot) will also entail an upward (i.e. negative) force of 1,200 tons, so that the total force will be

$$-189{\cdot}5-1{,}200 = -1{,}389{\cdot}5.$$

We shall need to keep account separately of the force on the constraint (which must be liquidated) and of the force on the elastic support (which need not). Denoting the former by $\mathbf{Y}_B$ and the latter as before by Y_B, we thus have to include a new operation in our Operations Table, namely,

$$\left.\begin{aligned}
&v_B = 1\text{:}\\
&\mathbf{Y}_B = -1{,}389{\cdot}5, && Y_A = 120, && Y_B = 1{,}200, && Y_C = 69{\cdot}5,\\
& && \mathbf{N}_A = 9{,}000, && \mathbf{N}_B = 2{,}750, && \mathbf{N}_C = -6{,}250.
\end{aligned}\right\} \quad \text{(iii)}$$

The other four operations entail r_A, r_B, r_C, r_D as before, and can be derived from 1 (*a*), 2 (*a*), 3 (*a*), 4 (*a*) of Table IV by increasing the moments and forces in the ratio by which B has been increased (i.e. by 250: cf. footnote to § 31); but the figures which previously appeared in the column headed Y_B must now appear in the column headed $\mathbf{Y}_B$. (By hypothesis the force on the elastic support is not affected by rotations.) Operations 1 (*a*), 2 (*a*), 3 (*a*), 4 (*a*) of Table VI

have been obtained in this manner; operation 5 (*a*) comes from (iii), and operations 1 (*b*), etc., are derived in the usual way.

32. We left the problem of § 26 having found that the pressures on the four supports, *assuming them all rigid*, are given by (5). Because under our new assumption the support at B yields under load, its force (= 184·2 tons) must initially be sustained by a rigid constraint, and it thus remains to investigate the effect of relaxing this constraint so that load is transferred in part to other supports. We can start the new Relaxation Table where the last left off—i.e. with actions given by line (*c*) of Table V (§ 28), except that the value given for Y_B now appears as $\mathbf{Y}_B$ and Y_B is now zero. Thus the first line is

	$\mathbf{N}_A$	$\mathbf{N}_B$	$\mathbf{N}_C$	$\mathbf{N}_D$	$\mathbf{Y}_B$	Y_A	Y_B	Y_C	Y_D
Initial actions	1	−0·25	−1	−0·5	184·20	71·71	0	141·69	46·40

Only $\mathbf{N}_A$, $\mathbf{N}_B$, $\mathbf{N}_C$, $\mathbf{N}_D$ and $\mathbf{Y}_B$ have to be liquidated.

The Relaxation Table is not reproduced, but is summarized in Table VII. The operations stated in the first column are the resultants of eighteen operations which appeared to reduce $\mathbf{Y}_B$ to zero and all moments to within $+1$; but line (*a*) shows that complete liquidation has not in fact been so nearly attained,—errors have accumulated in the last decimal place. It would, of course, be possible, starting with the figures in line (*a*), to carry relaxation farther; but the accuracy as it stands is amply sufficient for practical purposes.

33. Line (*c*) of Table VII gives, as before, the pressures on the supports. To the nearest 0·05 ton the figures are

$$Y_A = 76{\cdot}1, \quad Y_B = 173{\cdot}8, \quad Y_C = 150{\cdot}5_5, \quad Y_D = 43{\cdot}5_5 \text{ tons}, \qquad (8)$$

and comparing these with (5) of § 28 we see that depression of the support at B (through 0·2 foot approximately: see fifth line of first column in Table VII) has increased the pressures by amounts as under:

$$Y_A = +4{\cdot}4, \quad Y_B = -10{\cdot}4, \quad Y_C = +8{\cdot}8_5, \quad Y_D = -2{\cdot}8_5. \qquad (9)$$

Neglecting residual errors (i.e. taking these values as correct) we may say that the bending moments are altered on the same account by

$$\left.\begin{aligned} M_B &= 4{\cdot}4 \times 150 = +660, \\ M_C &= -2{\cdot}8_5 \times 150 = -427{\cdot}_5, \end{aligned}\right\} \qquad \text{(i)}$$

TABLE

No.	*Operation* $\times 10^{-3}$	$\mathbf{N}_A$	$\mathbf{N}_B$	$\mathbf{N}_C$	$\mathbf{N}_D$	$\mathbf{Y}_B$	Y_A	Y_B	Y_C	Y_D
1 (*a*)	$r_A = 1$	−900	−450	..	..	+9	−9	..	..	..
1 (*b*)	$r_A = 0{\cdot}1111$	−100	−50	..	..	+1	−1	..	..	..
2 (*a*)	$r_B = 1$	−450	−1,650	−375	..	+2·75	−9	..	+6·25	..
2 (*b*)	$r_B = 0{\cdot}06064$	−27·3	−100	−22·75	..	+0·167	−0·546	..	+0·379	..
3 (*a*)	$r_C = 1$	..	−375	−1,650	−450	−6·25	..	..	−2·75	+9
3 (*b*)	$r_C = 0{\cdot}06064$	..	$-22{\cdot}7_5$	−100	−27·3	−0·379	..	..	−0·167	+0·546
4 (*a*)	$r_D = 1$	..	..	−450	−900	..	..	..	−9	+9
4 (*b*)	$r_D = 0{\cdot}1111$	..	..	−50	−100	..	..	..	−1	+1
5 (*a*)	$v_B = 1$	+9	+2·75	−6·25	..	$-1{\cdot}389_5$	+0·12	+1·2	$+0{\cdot}069_5$	..
5 (*b*)	$v_B = 0{\cdot}72$	+6·48	+1·98	−4·50	..	−1	+0·086	+0·864	+0·05	..

† *N.B.*—As in preceding tables, bold symbols distinguish actions *on constraints*.

Operation and Multiplier (×10³)	N_A	N_B	N_C	N_D	Y_B	Y_A	Y_B	Y_C	Y_D
1 (*b*) ×13·07	−1,307	−653·5	..	..	+13·07	−13·07	..	..	..
2 (*b*) ×−0·18	+4·914	+18	+4·095	..	−0·03	+0·098	..	−0·068	..
3 (*b*) ×−10·45	..	+237·6	+1,045	+285·28	+3·96	..	..	+1·745	$-5{\cdot}70_5$
4 (*b*) ×2·87	..	..	−143·5	−287	..	..	..	−2·87	+2·87
5 (*b*) ×201·23	+1,303·97	+398·44	$-905{\cdot}53_5$	..	−201·23	+17·37	+173·8	+10·06	..
(*a*) Totals liquidated	+1·88	+0·54	+0·06	−1·72	−184·23	+4·40	+173·8	+8·87	−2·84
(*b*) Initial actions (§32)	1	−0·25	−1	−0·5	184·20	71·71	0	141·69	46·4
(*c*) Left on constraints	2·88	+0·29	−0·94	−2·22	−0·03	76·11	173·8	150·56	43·56

so that in place of (7) we now have values as under:

$$M_A = 0, \quad M_B = -2{,}084{\cdot}5, \quad M_C = -2{,}468{\cdot}5, \quad M_D = 0. \qquad (10)$$

This is the required solution.

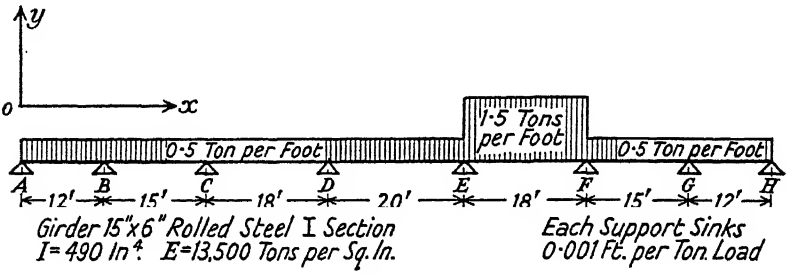

FIG. 4

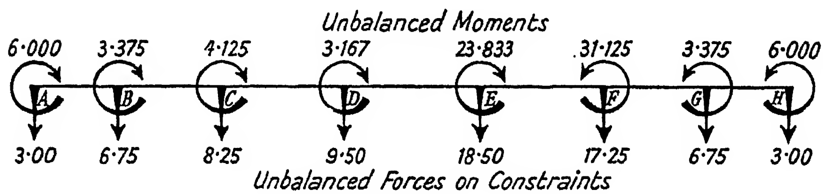

FIG. 5

Example

3. Fig. 4 shows a continuous girder of seven spans, resting on eight elastic supports.† Eliminating the distributed loadings as described in § 22, verify that the modified problem (§ 23) is as shown in Fig. 5, and calculate the reactions at the elastic supports.

[*Answer:* $R_A = 2{\cdot}41$, $R_B = 7{\cdot}03$, $R_C = 8{\cdot}434$, $R_D = 8{\cdot}834$, $R_E = 19{\cdot}47$, $R_F = 18{\cdot}376$, $R_G = 6{\cdot}036$, $R_H = 2{\cdot}41$ tons. Total 73·000 tons.]

CONTINUOUS BEAMS UNDER TRANSVERSE LOADING COMBINED WITH END THRUST

34. The wing spars of aeroplanes present a problem similar to those which have been treated in this chapter, but it is sometimes complicated by the circumstance that end thrust operates in conjunction with the transverse loading. On this account altered expressions hold for the clamping couples entailed when each span is treated separately in the manner of § 22, and our formulae (2) and (3) for the relevant influence coefficients also cease to apply; but apart from these numerical differences the methods of this chapter will apply without alteration, and accordingly it is only necessary to investigate

† This problem was treated by Hopkins (Ref. 3).

the effects of end thrust in the three problems which are indicated in Fig. 6.

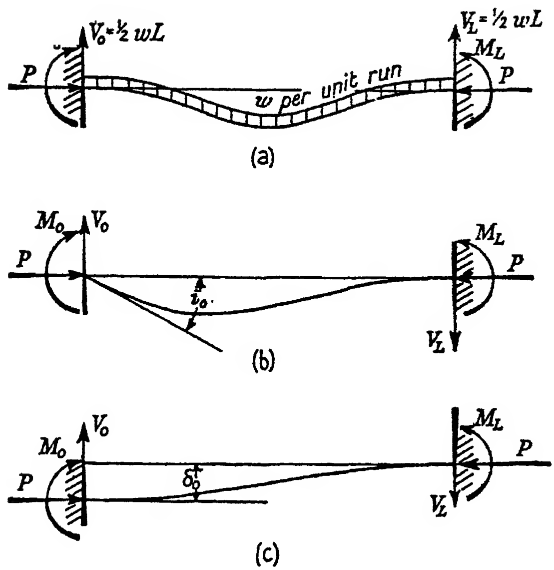

FIG. 6

35. Let M_S stand for the bending moment associated with the transverse loading under conditions of simple support at either end. Then, when allowance is made for the effect of the end thrust P acting on the distorted form, the equation which must be satisfied by the deflexion y is†

$$-By'' = M = Py + M_S - A - Cx \tag{11}$$

under conditions of terminal constraint, A and C being constants because the reactions and couples thereby brought into play will add a bending moment which must vary linearly from one end to the other. When the transverse loading w is uniform,

$$M_S = -\tfrac{1}{2}wx(x-L) \tag{i}$$

if the origin is taken at the left-hand end: then a particular solution of (11) is

$$y_P = \frac{1}{P}\left\{\tfrac{1}{2}wx(x-L) - w\frac{B}{P} + A + Cx\right\}, \tag{ii}$$

and the complementary function is

$$y_C = F\sin\alpha x + G\cos\alpha x \quad (\alpha^2 = P/B). \tag{iii}$$

The complete solution y has the form $y_P + y_C$.

† *Elasticity* § 205.

In problem (*a*) of Fig. 6 both ends are clamped on the line of thrust, w being operative; in problem (*b*), $w = 0$ and the right-hand end is clamped, the left-hand end supported, on the line of thrust; in problem (*c*), $w = 0$ and change of slope is prevented at both ends, the left-hand end being given a displacement away from the line of thrust. Thus in each of the three problems the constants A, C, F, G are defined by four conditions, and it is easy to verify that the requisite solutions are:

Problem (*a*)

$$y = \frac{1}{2}\frac{w}{P}\left[x(x-L)-\frac{L}{\alpha}\frac{\sin\alpha L-\sin\alpha x-\sin\alpha(L-x)}{1-\cos\alpha L}\right]; \tag{12}$$

Problem (*b*)

$$y = A_1\left[x-\frac{L\sin\alpha x}{\sin\alpha L}-\left(1-\frac{\alpha L\cos\alpha L}{\sin\alpha L}\right)\times\frac{\sin\alpha L-\sin\alpha x-\sin\alpha(L-x)}{\alpha(1-\cos\alpha L)}\right]$$

(A_1 being arbitrary); (13)

Problem (*c*)

$$y = A_2\left[1-\frac{x}{L}-\frac{1+\cos\alpha x-\cos\alpha L-\cos\alpha(L-x)}{\alpha L\sin\alpha L}\right]$$

(A_2 being arbitrary). (14)

36. The terminal bending moments according to (12) are given by

$$-M_0, -M_L = B(y_0'', y_L'') = \frac{wL^2}{4}\left[\frac{1-\beta\cot\beta}{\beta^2}\right] = \frac{wL^2}{12}F_1(\beta), \quad \text{say}, \tag{15}$$

where $$\beta = \tfrac{1}{2}\alpha L, \quad \text{so that } PL^2 = 4\beta^2 B. \tag{16}$$

The trend of $F_1(\beta)$ in the range $0 < 2\beta < 3$ is shown in Fig. 7.

37. According to (13) the slope (zero when $x = L$) is

$$i_0 = y_0' = A_1\left[2-\frac{\alpha L(1+\cos\alpha L)}{\sin\alpha L}\right] \tag{i}$$

when $x = 0$; and when A_1 has its value according to this equation the terminal bending moments are

$$\left.\begin{aligned} M_0 &= -By_0'' = \frac{Bi_0}{L}\frac{\beta(\sin 2\beta-2\beta\cos 2\beta)}{1-\cos 2\beta-\beta\sin 2\beta}, \\ M_L &= -By_L'' = -\frac{Bi_0}{L}\frac{\beta(2\beta-\sin 2\beta)}{1-\cos 2\beta-\beta\sin 2\beta}. \end{aligned}\right\} \tag{ii}$$

Therefore (by Statics) the terminal reactions are given by

$$V_0 = V_L = \frac{-M_0+M_L}{L} = -\frac{Bi_0}{L^2}\frac{2\beta^2\sin\beta}{\sin\beta-\beta\cos\beta} \tag{iii}$$

when the sign conventions conform with those of § 24.

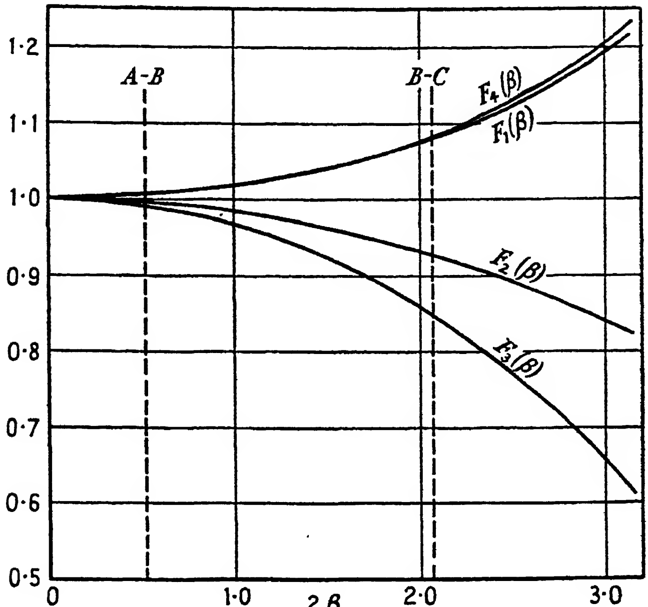

FIG. 7

Writing these results in the forms

$$V_0 = V_L = -6i_0\frac{B}{L^2}F_2(\beta), \qquad M_0 = 4i_0\frac{B}{L}F_3(\beta), \qquad M_L = -2i_0\frac{B}{L}F_4(\beta), \tag{17}$$

where

$$\left.\begin{aligned} F_2(\beta) &= \frac{\beta^2}{3}\frac{\sin\beta}{\sin\beta-\beta\cos\beta} = 1/F_1(\beta), \\ F_3(\beta) &= \frac{\beta}{4}\frac{\sin 2\beta-2\beta\cos 2\beta}{1-\cos 2\beta-\beta\sin 2\beta}, \\ F_4(\beta) &= \frac{\beta}{2}\frac{2\beta-\sin 2\beta}{1-\cos 2\beta-\beta\sin 2\beta}, \end{aligned}\right\} \quad 1 \text{ when } \beta\to 0, \tag{18}$$

we proceed to deduce expressions for the forces which come upon the constraints.

If as in § 25 l stands for the direction-cosine of the line drawn from the end which is moved to the end which is fixed, and if Y, N denote respectively a downward force and clockwise couple imposed on the constraint in virtue of a clockwise rotation r $(= i_0)$, then

$$V_0 = V_L = lY_M = -lY_F, \qquad -M_0 = N_M, \qquad M_L = N_F. \quad (19)$$

38. According to (14) the deflexion (zero when $x = L$) is

$$y_0 = A_2\left[1 - \frac{2(1-\cos\alpha L)}{\alpha L\sin\alpha L}\right] \quad \text{(i)}$$

when $x = 0$; and when A_2 has its value according to this equation

$$M_0 = -M_L = B(-y_0'', y_L'') = 2\frac{B}{L^2}y_0\frac{\beta^2\sin\beta}{\sin\beta - \beta\cos\beta}, \quad \text{(ii)}$$

and $$V_0 = V_L = \frac{-M_0+M_L}{L} = -4\frac{B}{L^3}y_0\frac{\beta^2\sin\beta}{\sin\beta-\beta\cos\beta}. \quad \text{(iii)}$$

These results may be written in the forms

$$V_0 = V_L = -12\frac{B}{L^3}y_0F_2(\beta), \qquad M_0 = -M_L = 6\frac{B}{L^2}y_0F_2(\beta), \quad (20)$$

where $F_2(\beta)$ has the significance given in the first of (18). Using the relations (19) we can deduce from them, as before, the actions imposed on the constraints.

39. Combining the results of §§ 37–8, and expressing them in the notation of influence coefficients (§ 25), we have finally

$$\begin{aligned} -\widehat{yy}_M &= \widehat{yy}_F = 12\frac{B}{L^3}F_2(\beta), \\ \widehat{rr}_M &= -4\frac{B}{L}F_3(\beta), \qquad \widehat{rr}_F = -2\frac{B}{L}F_4(\beta), \\ -\widehat{yr}_M &= \widehat{yr}_F = -\widehat{ry}_M = -\widehat{ry}_F = 6\frac{B}{L^2}lF_2(\beta), \end{aligned} \quad (21)$$

where $\beta^2 = PL^2/4B$ as before, and the trends of $F_2(\beta)$, $F_3(\beta)$, $F_4(\beta)$ in the range $0 < 2\beta < 3$ are as shown in Fig. 7. When P (and therefore β) is zero, all of $F_1(\beta)$, $F_2(\beta)$, $F_3(\beta)$ have the value unity, and then the formulae (21) reduce to those given in equations (2) and (3) of § 25.

Compared with the examples treated earlier in this chapter our problem is altered only in one particular, namely, that the influence

coefficients now have values depending on the magnitude of the end thrust P,—that is, on β as defined in (16). When P is specified, all influence coefficients have definite and constant values.

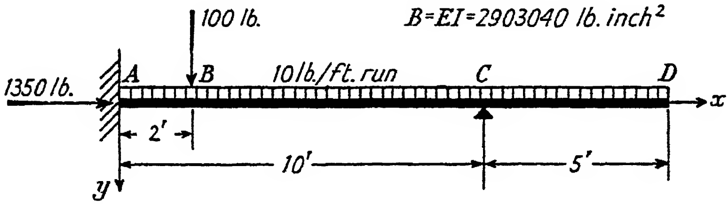

FIG. 8

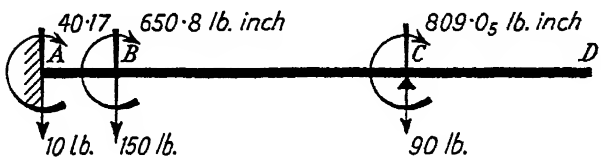

FIG. 9

An example

40. Fig. 8 presents an example of the problem as thus modified. One bay CD of the spar $ABCD$ is subjected to transverse forces only, but a thrust is imposed on the other bay AC, which also sustains a concentrated force in addition to the distributed loading. The spar is clamped at A and simply supported at C.

In treating this problem by relaxation methods we imagine that constraints are fitted at A, B and C which initially clamp those sections, and we first dispose of the distributed transverse loadings at the cost of terminal forces and moments imposed on the constraints. For the outer bay CD the requisite force and moment can be obtained by Statics, and the same is true of AB, BC as regards the forces; but the terminal moments on AB and BC (which are clamped at both ends, and which also sustain thrust) must be found from a special formula. This is problem (a) of Fig. 6, and the relevant formula is

$$M_0 = M_L = -\tfrac{1}{12}wL^2F_1(\beta), \qquad (15)\ bis$$

M_0, M_L being evidently negative, i.e. 'hogging', bending moments.

For AB, $2\beta = 0{\cdot}5175_5$ and accordingly $F_1(\beta) = 1{\cdot}004$; for BC, $2\beta = 2{\cdot}070$ and accordingly $F_1(\beta) = 1{\cdot}080$. Inserting the appropriate values of w and L, we find that

$$M_0 = M_L = -40{\cdot}17 \text{ lb. in. for } AB,$$
$$M_0 = -690{\cdot}9_5 \text{ lb. in. for } BC.$$

Disposing in this manner of the distributed loadings in all three bays, we reduce our problem to what is shown in Fig. 9. External forces and moments act (downward and clockwise) at A, B and C, given by

$$\left.\begin{aligned} \mathsf{Y}_A &= 10,\ \mathsf{Y}_B = 10+100+40 = 150,\ \mathsf{Y}_C = 40+50 = 90 \text{ (lb.)}, \\ \mathsf{N}_A &= 40\cdot17,\quad \mathsf{N}_B = 690\cdot9_5-40\cdot17 = 650\cdot8, \\ \mathsf{N}_C &= 1{,}500-690\cdot9_5 = 809\cdot0_5 \text{ (lb.-in. units)}. \end{aligned}\right\} \quad (22)$$

Initially they act on the constraints, but now the actions Y_B, N_B, N_C (*not* Y_A, Y_C, N_A, because the problem specifies *real* constraints at A and C which sustain these actions) must be transferred to the beam, or 'liquidated', by displacements of the types v_B, r_B, r_C (but *not* by v_C, v_A, r_A, since the data of the problem do not permit these displacements).

41. Evidently we require to know the effects on the constraints of displacements and of rotations imposed upon one end of a beam subjected to end thrust (problems (*b*) and (*c*) of Fig. 6). The requisite formulae are

$$\begin{aligned} -\widehat{yy}_M &= \widehat{yy}_F = 12\frac{B}{L^3}F_2(\beta), \\ \widehat{rr}_M &= -4\frac{B}{L}F_3(\beta), \qquad \widehat{rr}_F = -2\frac{B}{L}F_4(\beta), \\ -\widehat{ry}_M &= -\widehat{ry}_F = -\widehat{yr}_M = \widehat{yr}_F = 6\frac{B}{L^2}lF_2(\beta). \end{aligned} \quad (21)\,bis$$

Using (21) in relation to the members concerned, we easily find the effects of a unit displacement v_B and of unit rotations r_B, r_C. These we embody in a table of standard operations (Table VIII), and with this we proceed to liquidate Y_B, N_B, N_C as given in (22). The relaxation process is not here reproduced in full, but it is summarized in Table IX. From it the displacements of B and the rotations of B and C can be deduced, so that a solution can be effected of the problem in Fig. 9; and on replacing the transverse loadings with their associated reactions and clamping moments we have a complete solution of the problem with which we started in § 40.

42. Examples relating to spars in compression can be found in any text-book dealing with aircraft structures. As applications of the Relaxation Method they present no new feature, and for that reason none are given here.

TABLE VIII. *Standard Operations for Continuous Spar with End Thrust*

(Units: 1 lb. wt.; 1 inch; 1 radian.)

No. and nature of operation	*Actions to be liquidated*			*Actions not to be liquidated*		
	Y_B	N_B	N_C	Y_C	Y_A	N_A
1(*a*) $v_B = 1{\cdot}00$	*−2,545*	*28,359*	*−1,751*	*36*	*2,509*	*30,110*
1(*b*) $v_B = 0{\cdot}3929$	−1,000	11,143	−688	14	986	11,831
2(*a*) $r_B = 1{\cdot}00$	*28,359*	*−582,180*	*−65,451*	*1,751*	*−30,110*	*−243,072*
2(*b*) $r_B = 0{\cdot}1718$	4,871	−100,000	−11,242	301	−5,172	−41,752
3(*a*) $r_C = 1{\cdot}00$	*−1,751*	*−65,451*	*−102,612*	*1,751*	..	..
3(*b*) $r_C = 0{\cdot}9745$	−1,706	−63,785	−100,000	1,706	..	..

TABLE IX. *Summary of Relaxation Process for Continuous Spar with End Thrust*

Operation and total multiplier	Y_B	N_B	N_C	Y_C	Y_A	N_A
1(*b*) $\times 0{\cdot}4029$ ($v_B = 0{\cdot}4029 \times 0{\cdot}3929 = 0{\cdot}1583$)	−402·90	4,489·51	−277·20	5·64	397·26	4,766·71
2(*b*) $\times 0{\cdot}051718$ ($r_B = 0{\cdot}051718 \times 0{\cdot}1718 = 0{\cdot}008885$)	251·92	−5,171·80	−581·41	15·57	−267·49	−2,159·33
3(*b*) $\times -0{\cdot}000496$ ($r_C = -0{\cdot}000496 \times 0{\cdot}9745 = -0{\cdot}0004834$)	0·85	31·64	49·60	−0·81	..	..
Total actions	−150·13	−650·65	−809·01	20·40	129·77	2,607·38
Initial actions	150·00	650·78	809·03	90·00	10·00	40·17
Sum (= residual actions on constraints)	−0·13	0·13	0·02	110·40	139·77	2,647·55

In rarer instances a spar may be subjected to a tension T instead of the compression (P) which we have contemplated. Replacing P by $-T$ in (16), we have

$$4\gamma^2 B = TL^2, \quad \text{where } \gamma^2 = -\beta^2, \tag{23}$$

and hence, replacing β by $i\gamma$ in the formulae (15), (18) and (21), we can derive functions of γ (a real quantity when T is positive) to replace $F_1(\beta)$, $F_2(\beta)$, $F_3(\beta)$, $F_4(\beta)$. The new functions (for use in the case of tension) are

$$\left.\begin{aligned} F_1(\gamma) &= 3\frac{1-i\gamma\cot i\gamma}{-\gamma^2} = 3\frac{\gamma\coth\gamma-1}{\gamma^2},\\ F_2(\gamma) &= 1/F_1(\gamma) = \frac{1}{3}\frac{\gamma^2}{\gamma\coth\gamma-1},\\ F_3(\gamma) &= \frac{\gamma}{4}\frac{2\gamma\cosh 2\gamma-\sinh 2\gamma}{1-\cosh 2\gamma+\gamma\sinh 2\gamma},\\ F_4(\gamma) &= \frac{\gamma}{2}\frac{\sinh 2\gamma-2\gamma}{1-\cosh 2\gamma+\gamma\sinh 2\gamma}, \end{aligned}\right\} \tag{24}$$

γ being given by (23). Fig. 10 (corresponding with Fig. 7) shows the trends of $F_1(\gamma)$, etc., for values of 2γ between 0 and 3.†

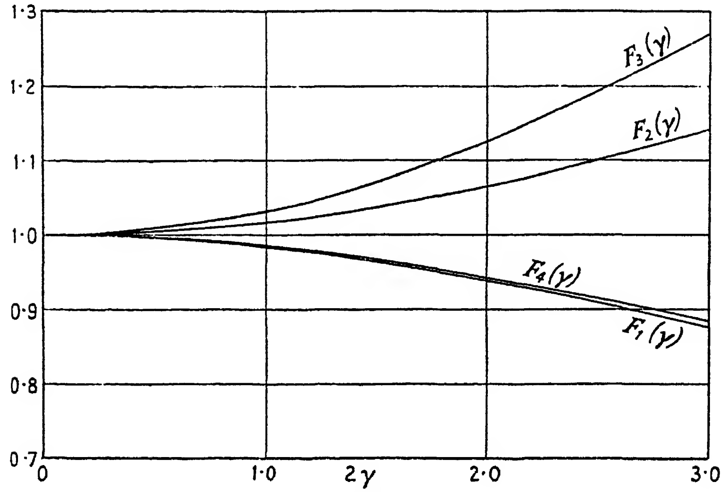

FIG. 10

RECAPITULATION

43. The stiff-jointed framework, as exemplified in this chapter by the special case of a continuous girder, presents a 'unit problem' which is slightly harder than that for pin-jointed frameworks, but when this has been solved its treatment by relaxation methods proceeds on the same lines as before. When end thrust or tension operates in conjunction with transverse loading the influence coefficients are modified; but their values can be found with the aid of diagrams (Fig. 7 or 10) or of tables showing the trends of certain functions, and thereafter the calculations are exactly similar.

The example of § 40 (Fig. 8) was solved by R. J. Atkinson and K. N. E. Bradfield (Ref. 1). Example 3 (Fig. 4) has been taken from a paper by H. J. Hopkins (Ref. 3).

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† Diagrams equivalent to Figs. 7 and 10 were given by D. Williams (Ref. 5, Figs. 3, 5 and 6). J. Morris (Ref. 4, p. 422) has shown the connexion between F_1, F_2, F_3, F_4 and the well-known 'Berry functions'. (Cf. Ref. 1, App. I.)

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III

RELAXATION METHODS APPLIED TO PLANE FRAMEWORKS HAVING RIGID JOINTS

44. A VERY slight extension of formulae obtained in §§ 24–5 (Chap. II) will bring within range of Relaxation Methods the important and difficult problems which are presented by plane frameworks having 'stiff' (i.e. rigid) joints. These problems fall into two classes, according as the applied forces act in directions parallel or perpendicular to the plane of the framework. The latter class we shall distinguish by the term **Grid Frameworks.**

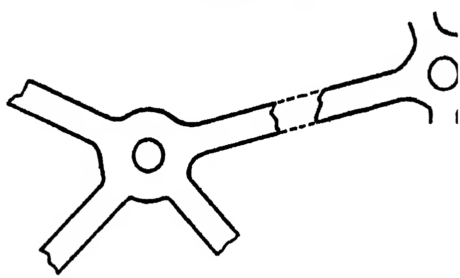

FIG. 11

Fig. 11 shows (diagrammatically) a typical joint in a plane stiff-jointed framework. Evidently the 'unit problem' consists, as in § 24, of a straight member having one end fixed and the other moved, but now the member must be allowed to have any orientation in the plane of the framework. Taking this as the (x, y) plane, we have to investigate the effects of displacements which in the first class of problem can comprise component translations **u** and **v** in the directions of Ox and Oy and rotations **r** about a perpendicular axis Oz, and in the second class of problem can comprise displacements **w** in the direction of Oz together with component rotations **p** and **q** about axes Ox and Oy. *Throughout this chapter we shall assume that every member is of uniform cross-section.*

I. PLANE FRAMEWORKS STRESSED BY FORCES IN THEIR OWN PLANES

The Unit Problem

45. In the first class of problem a member mf (Fig. 12) is inclined at any angle θ to Ox, and displacements **u**, **v**, **r** are imposed at m by which this end is moved to m'. Writing δ for the transverse displace-

ment and c for the contraction in length which results, we have from the diagram

$$\left.\begin{aligned}\delta &= \mathbf{v}\cos\theta - \mathbf{u}\sin\theta,\\ c &= \mathbf{u}\cos\theta + \mathbf{v}\sin\theta\end{aligned}\right. \tag{i}$$

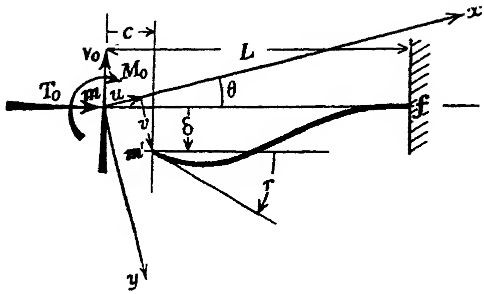

FIG. 12

Now we saw in § 24 that

$$\left.\begin{aligned}V_0 &= -6\frac{B}{L^3}(2\delta + L\mathbf{r}),\\ M_0 &= 2\frac{B}{L^2}(3\delta + 2L\mathbf{r})\end{aligned}\right. \tag{ii}$$

(B denoting the uniform flexural rigidity of the member) are the terminal shear force and moment which must be applied at m to maintain the deflexion δ and rotation $\mathbf{r}$; and to maintain the contraction c we must evidently apply a thrust T_0 given by

$$T_0 = EAc/L. \tag{iii}$$

Let X_M, Y_M, N_M, X_F, Y_F, N_F have the same significance as in Chapter II. Then from Fig. 12 it is clear that

$$\left.\begin{aligned}-X_M &= X_F = T_0\cos\theta + V_0\sin\theta,\\ -Y_M &= Y_F = T_0\sin\theta - V_0\cos\theta,\\ N_M &= -M_0, \qquad N_F = M_0 + V_0 L,\end{aligned}\right\} \tag{iv}$$

and on substituting in these expressions from (i)–(iii) we find that

$$\left.\begin{aligned}-X_M &= X_F = \frac{EA}{L}(l^2\mathbf{u} + lm\mathbf{v}) + \frac{6B}{L^3}(2m^2\mathbf{u} - 2lm\mathbf{v} - mL\mathbf{r}),\\ -Y_M &= Y_F = \frac{EA}{L}(lm\mathbf{u} + m^2\mathbf{v}) + \frac{6B}{L^3}(-2lm\mathbf{u} + 2l^2\mathbf{v} + lL\mathbf{r}),\\ N_M &= \frac{2B}{L^2}(3m\mathbf{u} - 3l\mathbf{v} - 2L\mathbf{r}),\\ N_F &= 2\frac{B}{L^2}(3m\mathbf{u} - 3l\mathbf{v} - L\mathbf{r}),\end{aligned}\right\} \tag{1}$$

l, m denoting $\cos\theta$, $\sin\theta$ respectively.

Influence coefficients

46. From (1) we can at once deduce expressions for the influence coefficients. They are

$$\left.\begin{aligned}
-\widehat{xx}_M &= \widehat{xx}_F = \frac{EA}{L}l^2+12\frac{B}{L^3}m^2 = 12\frac{B}{L^3}+Fl^2,\\
-\widehat{yx}_M &= \widehat{yx}_F = -\widehat{xy}_M = \widehat{xy}_F = \frac{EA}{L}lm-12\frac{B}{L^3}lm = Flm,\\
-\widehat{yy}_M &= \widehat{yy}_F = \frac{EA}{L}m^2+12\frac{B}{L^3}l^2 = 12\frac{B}{L^3}+Fm^2,\\
-\widehat{rx}_M &= -\widehat{rx}_F = -\widehat{xr}_M = \widehat{xr}_F = -6\frac{B}{L^2}m,\\
-\widehat{ry}_M &= -\widehat{ry}_F = -\widehat{yr}_M = \widehat{yr}_F = 6\frac{B}{L^2}l,\\
&\widehat{rr}_F = -2\frac{B}{L}, \qquad \widehat{rr}_M = -4\frac{B}{L},\\
\text{where}\qquad &F = \frac{EA}{L}-12\frac{B}{L^3}.
\end{aligned}\right\} \tag{2}$$

When $m = 0$ $(l = \pm 1)$ the expressions for $\widehat{yy}$, $\widehat{rr}$, $\widehat{ry}$ and $\widehat{yr}$ reduce (as they should) to what have been given already in (2) and (3) of Chapter II. When $B = 0$ (so that all flexural moments vanish) rotations have no effect and, since (cf. § 5)

$$l = \Delta x/L, \qquad m = \Delta y/L, \qquad EA/L^3 = \Omega, \tag{3}$$

the effects of displacement are given by

$$\left.\begin{aligned}
-\widehat{xx}_M &= \widehat{xx}_F = \Omega(\Delta x)^2,\\
-\widehat{yy}_M &= \widehat{yy}_F = \Omega(\Delta y)^2,\\
-\widehat{xy}_M &= \widehat{xy}_F = \Omega(\Delta x.\Delta y) = \widehat{yx}_F = -\widehat{yx}_M,
\end{aligned}\right\} \tag{4}$$

in conformity with (9) of § 6. Thus the formulae (2) can be applied to any member of uniform cross-section, whether its ends are both pin-jointed or both stiff-jointed. The cases in which one end is stiff-jointed but the other is hinged are left to the reader in the examples which follow. It is clear that actions on constraints will be entailed when a rotation is imposed at the stiff-jointed end, but not when it is imposed at the other.

Examples

1. Considering the member shown in Fig. 12, but now assuming that f (but not m) is pin-jointed, show that the expressions (2) are replaced in this instance by

$$\left.\begin{aligned}
-\widehat{xx}_M &= \widehat{xx}_F = \frac{EA}{L}l^2+\frac{3B}{L^3}m^2 = \frac{3B}{L^3}+Gl^2,\\
-\widehat{yy}_M &= \widehat{yy}_F = \frac{EA}{L}m^2+\frac{3B}{L^3}l^2 = \frac{3B}{L^3}+Gm^2,\\
-\widehat{yx}_M &= \widehat{yx}_F = -\widehat{xy}_M = \widehat{xy}_F = Glm,\\
\widehat{rx}_M &= \widehat{xr}_M = \frac{3B}{L^2}m = -\widehat{xr}_F, \qquad \widehat{rr}_M = -\frac{3B}{L},\\
\widehat{ry}_M &= \widehat{yr}_M = -\frac{3B}{L^2}l = -\widehat{yr}_F, \qquad \widehat{rr}_F = \widehat{rx}_F = \widehat{ry}_F = 0,\\
\text{where}\qquad G &= \frac{EA}{L}-\frac{3B}{L^3}.
\end{aligned}\right\} \quad (5)$$

2. Again considering the member shown in Fig. 12, but now assuming that m (but not f) is pin-jointed, show that the expressions (2) are replaced in this instance by

$$\left.\begin{aligned}
-\widehat{xx}_M &= \widehat{xx}_F = \frac{EA}{L}l^2+\frac{3B}{L^3}m^2 = \frac{3B}{L^3}+Gl^2,\\
-\widehat{yy}_M &= \widehat{yy}_F = \frac{EA}{L}m^2+\frac{3B}{L^3}l^2 = \frac{3B}{L^3}+Gm^2,\\
-\widehat{yx}_M &= \widehat{yx}_F = -\widehat{xy}_M = \widehat{xy}_F = Glm,\\
\widehat{rx}_M &= 0 = \widehat{ry}_M, \qquad \widehat{rx}_F = \frac{3B}{L^2}m, \qquad \widehat{ry}_F = -\frac{3B}{L^2}l,\\
\text{where}\qquad G &= \frac{EA}{L}-\frac{3B}{L^3}.
\end{aligned}\right\} \quad (6)$$

An example illustrative of the relaxation procedure

47. We shall use the formulae (2) in relation to a framework of which some joints are stiff but others free. Fig. 13 gives the dimensions and the nature of the external loading. The stiff members AB, BC, CD are each 50 inches long, with cross-sectional area 1 square inch and moment of inertia $I = 1{\cdot}25$ (inch)^4 units. AC is pin-jointed at both ends; its length is

$$50\sqrt{2} = 70{\cdot}7 \text{ inches},$$

and its cross-sectional area is 1·5 square inches. Thus 'B' for the members AB, BC, CD is 16,250 tons-inch² units, and 'EA' for the member AC is 19,500 tons.

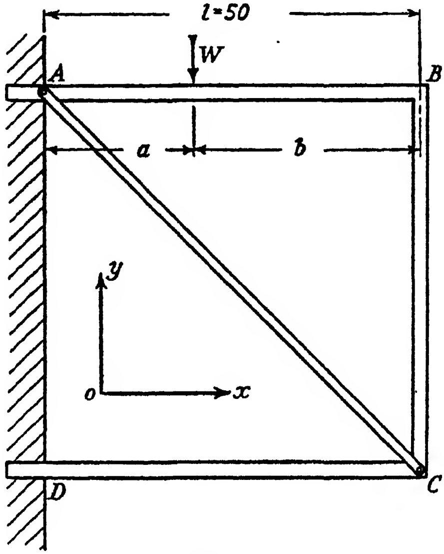

FIG. 13

We take axes Ox, Oy in the directions AB, DA, as shown in Fig. 13, and we adopt the ton-weight and inch as units of force and length. Then, considering the member AB and imagining B to be moved, we have as values for insertion in the formulae (2)

$$\left.\begin{aligned} &l = -1, \quad m = 0, \quad L = 50, \\ B = 16{,}250, \quad &B/L = 325, \quad B/L^2 = 6\cdot5, \quad 12B/L^3 = 1\cdot56, \\ &F = \frac{13{,}000}{50} - 12\frac{B}{L^3} = 260 - 1\cdot56 = 258\cdot44. \end{aligned}\right\} \tag{7}$$

Hence we may write (now using suffixes A for the fixed and B for the moving joint)

$$\left.\begin{aligned} &\widehat{xx}_A = 260, \quad \widehat{yy}_A = 1\cdot56, \quad \widehat{rr}_A = -650, \\ &\widehat{yx}_A = \widehat{xy}_A = \widehat{rx}_A = \widehat{xr}_A = 0, \\ &\widehat{ry}_A = -\widehat{yr}_A = 39 \\ \text{and}\quad & \\ &\widehat{xx}_B = -260, \quad \widehat{yy}_B = -1\cdot56, \quad \widehat{rr}_B = -1300, \\ &\widehat{yx}_B = \widehat{xy}_B = \widehat{rx}_B = \widehat{xr}_B = 0, \\ &\widehat{ry}_B = \widehat{yr}_B = 39. \end{aligned}\right\} \tag{8}$$

The other member attached to the joint B is BC, and the calculations for this member are exactly similar, except that $l = 0, m = -1$. So, for this member, the values (8) are replaced by

$$\left.\begin{aligned}
&\widehat{xx}_C = 1{\cdot}56, \quad \widehat{yy}_C = 260, \quad \widehat{rr}_C = -650,\\
&\widehat{yx}_C = \widehat{xy}_C = \widehat{yr}_C = 0,\\
&-\widehat{rx}_C = \widehat{xr}_C = 39,\\
\text{and}\qquad&\\
&\widehat{xx}_B = -1{\cdot}56, \quad \widehat{yy}_B = -260, \quad \widehat{rr}_B = -1{,}300,\\
&\widehat{yx}_B = \widehat{xy}_B = \widehat{ry}_B = \widehat{yr}_B = 0,\\
&\widehat{rx}_B = \widehat{xr}_B = -39.
\end{aligned}\right\} \quad (9)$$

These calculations replace (in the problem now under consideration) the simpler calculations for pin-jointed frameworks which were summarized (e.g.) in Table I, p. 7. Since different magnitudes, as well as different signs, now distinguish the influence coefficients appropriate to the fixed and moving joint of any member, tabulation is no longer worth while.

48. The calculations of § 8, on the other hand, are almost exactly reproduced, and here the results (namely, the actions on constraints which result from given displacements) may conveniently be tabulated. Thus, using (8) and (9), we find that a displacement u_B (in a relaxation confined to this particular joint and direction) involves forces and moments as under:

	forces in direction Ox		*forces in direction Oy*		*moments about Oz*
$u_B \times$	260 on A	$u_B \times$	0 on A	$u_B \times$	0 on A
	1·56 on C		0 on C		−39 on C
	−261·56 on B		0 on B		−39 on B

It will be seen that the sum of the *forces* imposed in either direction vanishes as it did in Chapter I (cf. § 9); but the sum of the *moments* does not vanish, being balanced (statically) by the forces ($\pm 1{\cdot}56$) which are imposed on B and C by shearing actions in BC.

These and similar results are recorded in the first line of Table X, which corresponds with Table II (p. 10) of Chapter I. The second line is derived in the manner described in § 9, and the next four lines (relating to the operations numbered 2 and 3) are obtained from (8) and (9) as described above.

Operations 4–6 are concerned with relaxations permitted to the joint C and affecting the members CB, CD, CA. For CB and CD the calculations are closely similar to those of § 47 (for CD we have $l = -1$, $m = 0$; for CB, $l = 0$, $m = 1$). In dealing with CA (which

TABLE X†

(Units: 1 ton weight; 1 inch.)

-Actions requiring to

Operation No.	*Nature of operation*	X_A	Y_A	N_A	X_B	Y_B	N_B
1 (*a*)	$u_B = 1$	*260*	*0*	*0*	*−261·56*	*0*	*−39*
1 (*b*)	$= 0{\cdot}3823$	99·4	0	0	−100	0	−14·9
2 (*a*)	$v_B = 1$	*0*	*1·56*	*39*	*0*	*−261·56*	*39*
2 (*b*)	$= 0{\cdot}3823$	0	0·60	14·9	0	−100	14·9
3 (*a*)	$r_B = 1$	*0*	*−39*	*−650*	*−39*	*39*	*−2,600*
3 (*b*)	$= 0{\cdot}3846$	0	−15	−250	−15	15	−1,000
4 (*a*)	$u_C = 1$	*138*	*−138*	*0*	*1·56*	*0*	*39*
4 (*b*)	$= 0{\cdot}2503$	34·6	−34·6	0	0·39	0	9·76
5 (*a*)	$v_C = 1$	*−138*	*138*	*0*	*0*	*260*	*0*
5 (*b*)	$= 0{\cdot}2503$	−34·6	34·6	0	0	65·1	0
6 (*a*)	$r_C = 1$	*0*	*0*	*0*	*−39*	*0*	*−650*
6 (*b*)	$= 0{\cdot}3846$	0	0	0	−15	0	−250
7 (*a*)	'Block displacement' $u_B = u_C = 1$	*398*	*−138*	*0*	*−260*	*0*	*0*
8 (*a*)	'Block displacement' $v_B = v_C = 1$	*−138*	*139·56*	*39*	*0*	*−1·56*	*39*

† Cf. the footnote to Table II (p. 10).

is pin-jointed at both ends) we must (cf. § 46) omit from (2) the terms which involve the flexural rigidity B. Then we have

$$F = \frac{EA}{L} = \frac{19{,}500}{50\sqrt{2}}$$
$$= 195\sqrt{2},$$
$$-l = m = 1/\sqrt{2},$$

and we find that all influence coefficients vanish except

$$\left.\begin{aligned} \widehat{xx}_A = \widehat{yy}_A &= 138, \\ \widehat{yx}_A = \widehat{xy}_A &= -138, \\ \text{and} \qquad \widehat{xx}_C = \widehat{yy}_C &= -138, \\ \widehat{yx}_C = \widehat{xy}_C &= 138. \end{aligned}\right\} \tag{10}$$

Operations 7 and 8, Table X, are combinations respectively of Operations 1 and 4 and of Operations 2 and 5, which later we shall find convenient. The term 'block displacement' will be explained in Chapter IV.

TABLE X

(Units: 1 ton weight; 1 inch.)

be liquidated

X_C	Y_C	N_C	X_D	Y_D	N_D	*Resultant actions on 'block' BC*
1·56	*0*	*−39*	*0*	*0*	*0*	$X = X_B+X_C = -260$, $Y = Y_B+Y_C = 0$, $N = N_B+N_C-50X_B = +13{,}000$
0·60	0	−14·9	0	0	0	
0	*260*	*0*	*0*	*0*	*0*	
0	99·4	0	0	0	0	
39	*0*	*−650*	*0*	*0*	*0*	
15	0	−250	0	0	0	
−399·56	*138*	*39*	*260*	*0*	*0*	
−100	34·6	9·76	65·1	0	0	
138	***−399·56***	*39*	*0*	*1·56*	*39*	
34·6	**−100**	9·76	0	0·39	9·76	
39	*39*	***−2,600***	*0*	*−39*	*−650*	
15	15	**−1,000**	0	−15	−250	
−398	*138*	*0*	*260*	*0*	*0*	$X = X_B+X_C = -658$, $Y = Y_B+Y_C = 138$, $N = +13{,}000$.
138	*−139·56*	*39*	*0*	*1·56*	*39*	$X = X_B+X_C = +138$, $Y = Y_B+Y_C = -141{\cdot}12$, $N = +78$.

Treatment of a transversely loaded member

49. Reverting to Fig. 13, we begin by disposing (in the manner of § 22, Chap. II) of the transverse load which acts near the middle of the stiff member AB, thus reducing our problem to one in which applied forces and moments act only at sections supported by constraints. Treating AB as a girder separated from the framework and having both its ends clamped rigidly, we can by ordinary methods determine the values of R_1, R_2, M_1, M_2 in Fig. 14. With symbols as given in that diagram, we easily find that

$$\begin{aligned} R_1 l^3 &= Wb^2(3a+b), & R_2 l^3 &= Wa^2(a+3b), \\ M_1 l^2 &= Wab^2, & M_2 l^2 &= Wa^2b, \end{aligned} \tag{11}$$

and assuming that $a = 20$ inches, $b = 30$ inches, $W = 1{,}000$ tons weight, we have in this instance

$$\left.\begin{array}{ll} R_1 = 648 \text{ tons}, & R_2 = 352 \text{ tons}, \\ M_1 = 7{,}200 \text{ tons-inches}, & M_2 = 4{,}800 \text{ tons-inches}. \end{array}\right\} \quad (12)$$

These actions combined with W constitute a self-equilibrating system which, by the Reciprocal Theorem (*Elasticity* § 12), will do no work in any displacement or distortion of AB which can occur as the result of actions applied to its ends:† therefore, as concerns the rest of the framework, the member AB *with* W, R_1, R_2, M_1, M_2 *operative* behaves exactly as though it were unloaded. Accordingly we imagine these forces to be applied to the member, and forces and moments equal and opposite to R_1, R_2, M_1, M_2 to be applied at the same time to the joints at A and B. Since the new forces cancel identically we have made no difference to the problem; during the process of relaxation we can neglect the actions on AB; and hence (for the purpose of this process) our problem is reduced to one in which the loads are applied only at the joints.

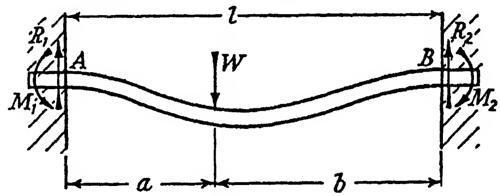

FIG. 14

The modified problem

50. Initially these loads come wholly upon the constraints, and at A and D (since no movement is permitted) they will remain on the constraints: therefore in the process of relaxation we have to 'liquidate' only the actions imposed on the constraints at B and C. Thus our problem is reduced (for immediate purposes) to what is shown in Fig. 15. We have initially (in tons-inch units)

$$\mathrm{X}_B = 0, \qquad \mathrm{Y}_B = -352, \qquad \mathrm{N}_B = +4{,}800. \qquad (13)$$

The relaxation procedure

We now proceed to liquidate these actions, i.e. (§ 4) to transfer them from the constraints to the framework. The process is in

† According to the theorem such work must equal the work done by these terminal actions in the actual displacements of Fig. 14; and the latter work is zero, since the distortion of Fig. 14 involves no relative movement and no rotation of the terminal sections.

essence similar to what was described in §§ 13–14 of Chapter I, but now the relaxations are conveniently performed by 'stages' in which, alternately, either joint-displacements unaccompanied by rotation are permitted, or joint-rotations unaccompanied by displacement. Table XI gives the whole sequence, which consists of four stages (ten operations in all) and reduces the unliquidated forces and moments,

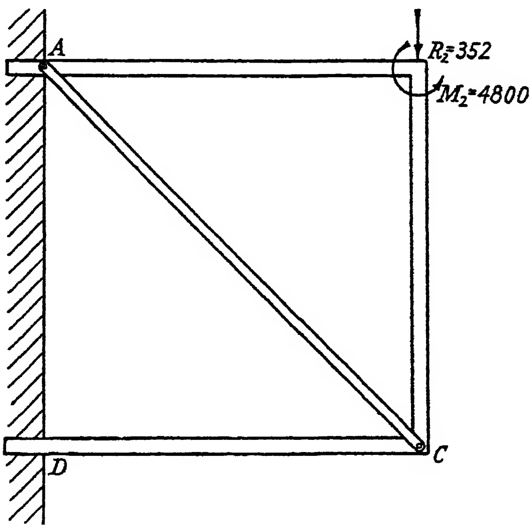

FIG. 15

in this example, to 0·4 per cent. and 0·05 per cent., respectively, of their initial values. Such accuracy is, of course, more than sufficient for practical purposes, and greater accuracy could be attained if required, at the cost of slightly greater labour.†

51. Some detailed description may be helpful to a clearer understanding of the liquidation procedure as applied in this example.

Stage I. Since the procedure works most rapidly as applied to moments, we begin by disposing of N_B. Referring to Table X we see that Operation 3 will dispose of 16 units of N_B at the cost of introducing a moment N_C of -4 units, and that Operation 6 will dispose of this latter moment by introducing similarly a moment N_B of $+1$ unit. The double operation disposes of 15 units of N_B

† Compared with the exact solution as obtained by orthodox methods (§ 55), our approximate solution (even when stopped after this small number of operations) has given the actions in the constituent members with a maximum error of about 1·7 per cent.

while leaving N_C unaltered: consequently to cancel the moment $N_B = 4{,}800$ in line 1 of Table XI we must by Operation I (i) introduce a moment at B of magnitude

$$-\tfrac{16}{15}\times 4{,}800 = -5{,}120,$$

leaving a moment of -320 to be cancelled by Operation I (ii).

Table XI

(Brackets indicate the action which the next

Actions requiring to

Stage	Operation and multiplier	X_A	Y_A	N_A	X_B	Y_B	N_B
	(Initial forces and moments)	..	..	..	0	−352	(4,800)
I (i)	3 (*b*) ×5·12	0	−76·8	−1,280	−76·8	76·8	−5,120
		0	−76·8	−1,280	−76·8	−275·2	−320
I (ii)	6 (*b*) × −1·28	0	0	0	19·2	0	320
		0	−76·8	−1,280	−57·6	−275·2	0
II (i)	8 (*a*) × −2·96	408·5	−413	−115·4	0	4·6	−115·4
		408·5	−489·8	−1,395·4	−57·6	−270·6	−115·4
II (ii)	7 (*a*) × −0·89	−354·2	122·9	0	231·5	0	0
		54·3	−366·9	−1,395·4	173·9	−270·6	−115·4
II (iii)	1 (*a*) ×0·69	179·4	0	0	−180·5	0	−26·9
		233·7	−366·9	−1,395·4	−6·6	(−270·6)	−142·3
II (iv)	2 (*a*) × −1	0	−1·6	−39	0	261·6	−39
		233·7	−368·5	−1,434·4	−6·6	−9	(−181·3)
III (i)	3 (*b*) × −0·16	0	2·4	40	2·4	−2·4	160
		233·7	−366·1	−1,394·4	−4·2	−11·4	−21·3
III (ii)	6 (*b*) × −0·1	0	0	0	1·5	0	25
		233·7	−366·1	−1,394·4	−2·7	(−11·4)	3·7
IV (i)	2 (*b*) × −0·1	0	−0·1	−1·5	0	10	−1·5
		233·7	−366·2	−1,395·9	(−2·7)	−1·4	2·2
IV (ii)	1 (*a*) × −0·01	−2·6	0	0	2·6	0	0·4
	Totals for Fig. 15	231·1	−366·2	−1,395·9	−0·1	−1·4	2·6
	Added in § 54	0	−648	−7,200			
	Totals for Fig. 13	231·1	−1,014·2	−8,595·9			

Stage II. We now start to liquidate *forces*, and we begin this process with operations (Nos. 7 and 8) involving more than one joint. Time will be saved if the required displacements are determined at once, and precisely (from simultaneous equations), to cancel resultant forces $\mathbf{X} = 0$, $\mathbf{Y} = -294{\cdot}4$ tons and a resultant moment $\mathbf{N} = 2{,}880$ tons-inches on BC by a combination of Operations nos.

1 (*a*), 7 (*a*) and 8 (*a*). Let k_1, k_7, k_8 be the required multipliers: then from the figures given in the last column of Table X we have

$$-260k_1 - 658k_7 + 138k_8 = 0,$$
$$138k_7 - 141{\cdot}12k_8 = 294{\cdot}4,$$
$$13{,}000k_1 + 13{,}000k_7 + 78k_8 = -2{,}880,$$

TABLE XI

following operation was intended to liquidate.)

be liquidated

X_C	Y_C	N_C	X_D	Y_D	N_D	*Resultants for 'block' BC*
..	..	..	..	..	..	
76·8	0	−1,280	0	0	0	
76·8	0	(−1,280)	0	0	0	
−19·2	−19·2	1,280	0	19·2	320	
57·6	−19·2	0	0	19·2	320	$X = X_B + X_C = 0$ $Y = Y_B + Y_C = -294{\cdot}4$ $N = 2880$
−408·5	413	−115·4	0	−4·6	−115·4	
−350·9	393·8	−115·4	0	14·6	204·6	
354·2	−122·9	0	−231·5	0	0	
3·3	270·9	−115·4	−231·5	14·6	204·6	
1·1	0	−26·9	0	0	0	
4·4	(270·9)	−142·3	−231·5	14·6	204·6	
0	−260	0	0	0	0	
4·4	10·9	−142·3	−231·5	14·6	204·6	
−2·4	0	40	0	0	0	
2·0	10·9	(−102·3)	−231·5	14·6	204·6	
−1·5	−1·5	100	0	1·5	25	
0·5	9·4	−2·3	−231·5	16·1	229·6	
0	−9·9	0	0	0	0	
0·5	−0·5	−2·3	−231·5	16·1	229·6	
−0·02	0	0·4	0	0	0	
0·5	−0·5	−1·9	−231·5	16·1	229·6	

and these equations are satisfied (approximately) by the values

$$k_1 = 0{\cdot}69, \qquad k_7 = -0{\cdot}89, \qquad k_8 = -2{\cdot}96. \tag{14}$$

The multipliers for the first three operations of Stage II are chosen accordingly. After those operations the only large *forces* remaining are Y_B, Y_C, and these are reduced simultaneously by an

operation of the type 2(*a*). We end Stage II at this point, and proceed to liquidate the unbalanced moments at *B* and *C*.

Stage III. Two operations serve to eliminate the greater part of either moment, and we turn again to the forces.

Stage IV. Two operations reduce the largest residual force from 11·4 to 1·4 tons and at the same time reduce the larger of the unbalanced moments from 3·7 to 2·6 tons-inches. Comparing these figures with what we had given initially, we see that they represent about 0·4 per cent. of the specified force and 0·05 per cent. of the specified moment. It is not worth while (although it would be easy) to proceed to a closer approximation.

Calculation of displacements and rotations

52. When its approximation is deemed sufficient, we complete the solution by finding the displacements and the stresses which it entails. The former are deduced as shown in Table XII, where multipliers taken from the second column of Table XI are combined with displacements taken from the second column of Table X.

TABLE XII

Stage	u_B	v_B	r_B	u_C	v_C	r_C
I (i)	..	..	5·12×0·3846	..	..	..
I (ii)	..	..	..	..	..	−1·28×0·3846
II (i)	..	−2·96	..	..	−2·96	..
II (ii)	−0·89	..	..	−0·89	..	..
II (iii)	0·69	..	..	..	..	..
II (iv)	..	−1·00	..	..	..	..
III (i)	..	..	−0·16×0·3846	..	..	..
III (ii)	..	..	..	..	..	−0·1×0·3846
IV (i)	..	−0·1×0·3823	..	..	..	..
IV (ii)	−0·01	..	..	..	..	..
Totals .	−0·210	−3·998	$1·907_5$	−0·890	−2·960	$−0·530_5$

Calculation of the actions in constituent members

53. The actions which in our solution are imposed on the constraints at *A* and *D* can be read off from the last line of Table XI, and the actions in the constituent members can be deduced from the known displacements with the aid of (8), (9), etc. Thus we have from (8), substituting for the displacement u_B from Table XII,

$$\left.\begin{aligned}
&\text{tension in } AB = \widehat{xx}_A \times u_B = 260\times -0{\cdot}210 = -54{\cdot}6 \text{ tons,}\\
&\text{and we find similarly that}\\
&\text{tension in } BC = 260\times(v_B - v_C) = 260\times -1{\cdot}038\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -269{\cdot}9 \text{ tons,}\\
&\text{tension in } CD = 260\times u_C = 260\times -0{\cdot}890 = -231{\cdot}4 \text{ tons,}\\
&\text{tension in } AC = \surd 2\times 138(u_C - v_C) = 195\times 2{\cdot}07\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 403{\cdot}6_5 \text{ tons.}
\end{aligned}\right\}\quad (15)$$

In this simple example we can proceed at once to calculate the shearing actions by Statics, since we know (from Table XI) the forces on the joints and (from the above figures) how much of each force is imposed by tensions. We find in this way that†

$$\left.\begin{aligned}
&\text{shear in } AB \text{ (from force on } B) = (352 - 1{\cdot}4) - 269{\cdot}9\\
&\qquad\qquad\qquad\qquad\qquad\qquad = 80{\cdot}7 \text{ tons,}\\
&\text{shear in } BC \text{ (from force on } B) = 54{\cdot}6 + 0{\cdot}1 = 54{\cdot}7 \text{ tons,}\\
&\text{shear in } CD \text{ (from force on } C) = 1/\surd 2\times 403{\cdot}6_5 - 269{\cdot}9 + 0{\cdot}5\\
&\qquad\qquad\qquad\qquad\qquad\qquad = 16{\cdot}0 \text{ tons,}
\end{aligned}\right\}\quad (16)$$

the senses of the shearing forces being such as would rotate AB, BC, CD respectively clockwise, clockwise, and counter-clockwise. The bending moments in AB at A and B are given by

$$\left.\begin{aligned}
&M_A = \widehat{ry}_A v_B + \widehat{rr}_A r_B = 39\times(-3{\cdot}998) - 650\times 1{\cdot}907_5,\\
&\qquad\qquad\qquad\qquad = -155{\cdot}9 - 1{,}240 = -1{,}396 \text{ tons-inches,}\\
&M_B = \widehat{ry}_B v_B + \widehat{rr}_B r_B = 39\times(-3{\cdot}998) - 1{,}300\times 1{\cdot}907_5,\\
&\qquad\qquad\qquad\qquad = -155{\cdot}9 - 2{,}480 = -2{,}636 \text{ tons-inches,}\\
&\text{and for } CD \text{ we have similarly}\\
&M_D = 39v_C - 650r_C \quad = 39\times(-2{\cdot}96) + 650\times 0{\cdot}530_5\\
&\qquad\qquad\qquad\qquad = -115{\cdot}4 + 344{\cdot}6 = 229 \text{ tons-inches,}\\
&M_C = 39v_C - 1{,}300r_C = 39\times(-2{\cdot}96) + 1{,}300\times 0{\cdot}530_5\\
&\qquad\qquad\qquad\qquad = -115{\cdot}4 + 689{\cdot}2 = 574 \text{ tons-inches.}
\end{aligned}\right\}\quad (17)$$

Allowing for the unbalanced couples which our solution has left on the constraints at B and C, we deduce that the terminal bending moments for BC are

$$\left.\begin{aligned}
M_B &= (4{,}800 - 2{\cdot}6) - 2{,}636 = 2{,}161{\cdot}_5 \text{ tons-inches,}\\
M_C &= 574 + 1{\cdot}9 \qquad\qquad\quad = 576 \text{ tons-inches.}
\end{aligned}\right\}\quad (18)$$

† Allowance is made in these expressions for the forces still remaining on the constraints according to Table XI.

These values accord well with (16) and with the residual actions given in the last line of Table XI. They are compared in § 55 with 'exact' values (entailing no residual actions on the constraints) which are there calculated by orthodox methods.

Completion of the solution

54. We have completed our solution of the modified problem presented in Fig. 15, and it only remains to complete the solution

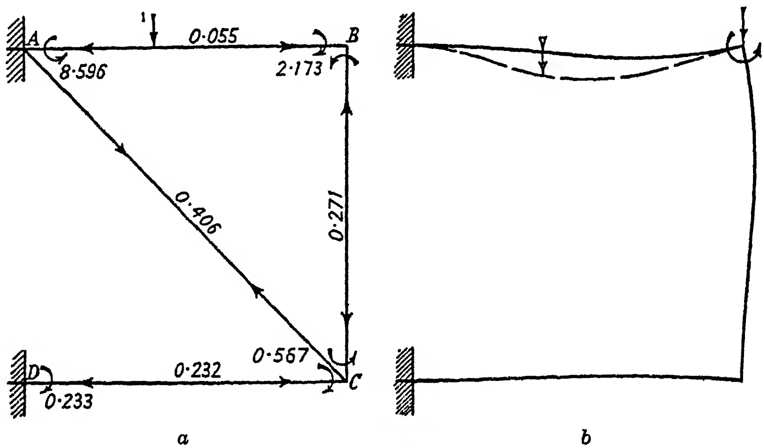

FIG. 16

for Fig. 13. We determined the values of R_1, R_2, M_1, M_2 (Fig. 14) in § 49, and we have already applied R_2 and M_2 to the framework as $\mathbf{Y}_B$, $\mathbf{N}_B$): if, then, we now apply R_1, M_1 to the constraint at A (as in the supplementary line of Table XI), we have a solution satisfying all the conditions which are imposed when the framework is loaded by a force of 1,000 tons applied to AB in the manner of Fig. 13, and we can construct Fig. 16 as a diagrammatic summary of results.

It is perhaps worth while to remark that a load of 1,000 units was imposed for numerical convenience; the value assumed is immaterial in a problem governed by Hooke's law. In Fig. 16 *a* values are given for the tensions and thrusts in tons, and for the bending moments in tons-inches, which result from each ton of applied load W (Fig. 13). In Fig. 16 *b* the full lines show (greatly exaggerated) the distortions produced by the applied force and moment in the simplified problem of Fig. 15; dotted lines show the additional distortion of AB under the actions of Fig. 14, thus giving the total distortion of the framework in the problem taken initially (Fig. 13).

Examples relating to stiff-jointed frames will be given in Chapter IV, following an explanation of the use of 'block' and 'group' displacements.

Comparable solution by conventional methods

55. Although the Relaxation Method has the valuable feature that it is self-checking in the sense that we can judge its accuracy

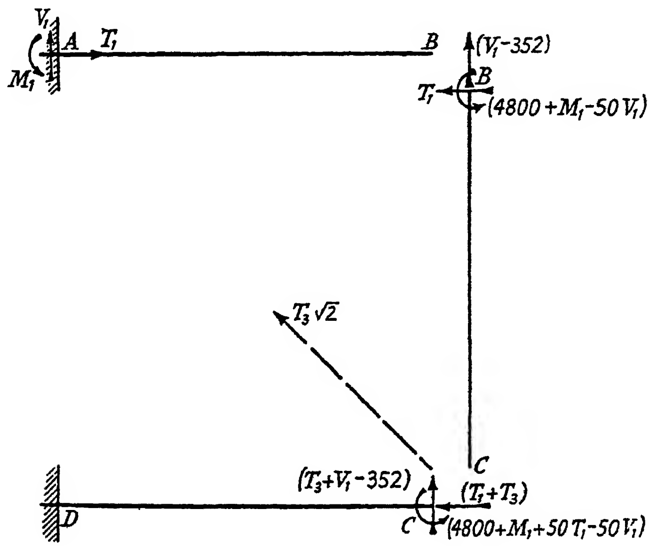

FIG. 17

from the actions left unliquidated, nevertheless it will be instructive to compare its results for this problem with those obtained by the conventional 'Method of Least Work'.

The order of redundancy of this framework is 4. We may take as unknowns the tensions in AB and AC (which we denote by T_1 and $T_3\sqrt{2}$), the bending moment (M_1) which is applied to AB at A, and the shear (V_1) in AB: then the actions in the remaining members can be deduced from statical considerations, and are as indicated in Fig. 17.

In AB, the tension is T_1,
,, the bending moment M ranges from M_1 to (M_1-50V_1);
in BC, the tension is (V_1-352),
M ranges from $(4,800+M_1-50V_1)$ to $(4,800+M_1+50T_1-50V_1)$;
in CD, the thrust is (T_1+T_3),
M ranges from $(4,800+M_1+50T_1-50V_1)$ to $(M_1-12,800+50T_1+50T_3)$.

Now for any member of length L, if the tension is P and if the bending moment ranges from M' to M'', it is easy to show that

$$\text{strain-energy of direct tension} = \frac{1}{2}\frac{P^2L}{EA},$$

$$\text{strain-energy of flexure} = \frac{1}{6}\frac{L}{B}(M'^2+M'M''+M''^2).$$

Therefore, since for AB, BC, CD,

$$L = 50, \qquad EA = 13{,}000, \qquad B = 16{,}250,$$

and for AC,

$$L = 50\sqrt{2}, \qquad EA = 19{,}500, \qquad B = 0 \text{ (in effect)},$$

the total strain-energy of direct thrust and tension is

$$U_t = \frac{50}{2\times 13{,}000}\left[T_1^2+(V_1-352)^2+(T_1+T_3)^2+\frac{2\sqrt{2}}{1\cdot 5}T_3^2\right]$$

and the total strain-energy of flexure is

$$U_f = \frac{50}{97{,}500}[M_1^2+M_1(M_1-50V_1)+(M_1-50V_1)^2+$$
$$+(\text{similar terms for } BC \text{ and } CD)].$$

The total strain-energy $U = U_t+U_f$. Accordingly we have

$$\tfrac{3}{5}\times 13{,}000U$$
$$= 15[T_1^2+(V_1-352)^2+(T_1+T_3)^2+\frac{2}{1\cdot 5}\sqrt{2}\,T_3^2+$$
$$+4M_1^2+M_1(M_1-50V_1)+(M_1-50V_1)^2+$$
$$+(4{,}800+M_1-50V_1)^2+(4{,}800+M_1-50V_1)(4{,}800+M_1+50T_1-50V_1)+$$
$$+2(4{,}800+M_1+50T_1-50V_1)^2+$$
$$+(2M_1-8{,}000+100T_1+50T_3-50V_1)(M_1-12{,}800+50T_1+50T_3)],$$

and hence (since U assumes its minimum value in accordance with Castigliano's 'Principle of Least Work') we deduce that

$$\frac{\partial U}{\partial T_3} = 0; \quad \text{therefore } 3M_1-50V_1+150\cdot 15T_1+100\cdot 433T_3 = 20{,}800; \qquad \text{(i)}$$

$$\frac{\partial U}{\partial T_1} = 0; \quad \text{therefore } 9M_1-300V_1+400\cdot 3T_1+150\cdot 15T_3 = 9{,}600; \qquad \text{(ii)}$$

$$\frac{\partial U}{\partial V_1} = 0; \quad \text{therefore } 12M_1-500\cdot 15V_1+300T_1+50T_3 = -25{,}652\cdot 8; \qquad \text{(iii)}$$

$$\frac{\partial U}{\partial M_1} = 0; \quad \text{therefore } 18M_1-600V_1+450T_1+150T_3 = -4{,}800. \qquad \text{(iv)}$$

Solving the simultaneous equations (i)–(iv), we have, approximately,

$$T_1 = -54\cdot 8, \qquad T_3 = 287\cdot 3,$$
$$V_1 = 80\cdot 7, \qquad M_1 = 1{,}408,$$

and we can deduce the other actions shown in Fig. 17. We find that

Tension in $AB = T_1$	$= -54{\cdot}8$ (tons)	$(-54{\cdot}6)$
Tension in $BC = V_1-352$	$= -271{\cdot}3$ (tons)	$(-269{\cdot}9)$
Tension in $CD = -(T_1+T_3)$	$= -232{\cdot}5$ (tons)	$(-231{\cdot}4)$
Tension in $AC = T_3\sqrt{2}$	$= 406{\cdot}2$ (tons)	$(403{\cdot}65)$
Shear in $AB = V_1$	$= 80{\cdot}7$ tons	$(80{\cdot}7)$
Shear in $BC = -T_1$	$= 54{\cdot}8$ tons	$(54{\cdot}7)$
Shear in $CD = T_3+V_1-352$	$= 16$ tons	$(16{\cdot}0)$

Bending moments:—

in AB:	$M_A = M_1 = 1{,}408$;	$(1{,}396)$
	$M_B = 50V_1-M_1 = 2{,}627$ tons-inches	$(2{,}636)$
in BC:	$M_B = 4{,}800+M_1-50V_1 = 2{,}173$;	$(2{,}161{\cdot}_5)$
	$M_C = M_B+50T_1 = 567$ tons-inches	(576)
in CD:	$M_C = 567$;	(574)
	$M_D = 800-567 = 233$ tons-inches	(229)

The figures in brackets give (for comparison) the values obtained in this chapter by Relaxation Methods.

II. GRID FRAMEWORKS STRESSED BY TRANSVERSE FORCES

The unit problem

56. In the second class of problem, as distinguished in § 44, the joint displacement has no component in the plane of the framework, consequently (to the first order of small quantities) members experience no change in length, therefore no tension or compression. But a joint may now rotate about any axis in the plane of the framework, and on this account, in the 'unit problem' appropriate to a grid framework, we must examine the effects of imposing on one end of a member mf (Fig. 18) rotations $\mathbf{p}$ and $\mathbf{q}$ about axes through m parallel to Ox and Oy, also a displacement $\mathbf{w}$ in the direction of Oz.†

Writing l for $\cos\theta$ and m for $\sin\theta$, where θ is the angle (in the (x, y) plane) which mf makes with the axis Ox, we may resolve $\mathbf{p}$ and $\mathbf{q}$ to find the rotations β and τ about the perpendicular axes mB and mf. The vector law gives

$$\beta = m\mathbf{p}-l\mathbf{q}, \qquad \tau = l\mathbf{p}+m\mathbf{q}.$$

Evidently β is a rotation, $\mathbf{w}$ a displacement, by which the member is bent in a plane perpendicular to xOy; so these quantities replace $\mathbf{r}$ and δ, respectively, in the formulae (ii) of § 45, V_0 now denoting a force in the direction zO applied to the member at m, and M_0 a

† The axes of the rotations $\mathbf{p}$, $\mathbf{q}$, β, τ are indicated in Fig. 18 by double-headed arrows.

couple having axis mB and also applied at m. In consequence of the rotation τ the member is twisted in a right-handed direction about mf; so a torque T_0 is entailed at m, right-handed about mf and of magnitude

$$T_0 = C\tau/L,$$

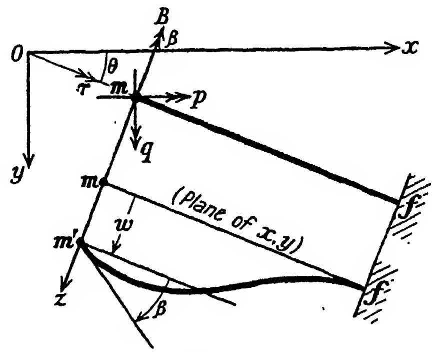

FIG. 18

where C is the 'torsional rigidity' of the (uniform) cross-section (*Elasticity* § 339). The forces exerted on the constraints at m and f will thus be given by

$$-Z_F = Z_M = V_0 = -6\frac{B}{L^3}(2\mathrm{w}+L\beta), \tag{i}$$

B standing as before for the (uniform) flexural rigidity of the member;† the couples exerted on the constraints at the end which is moved will be given by

$$\left.\begin{aligned} L_M &= -(lT_0+mM_0) = -l\frac{C}{L}\tau-2m\frac{B}{L^2}(3\mathrm{w}+2L\beta), \\ M_M &= -(mT_0-lM_0) = -m\frac{C}{L}\tau+2l\frac{B}{L^2}(3\mathrm{w}+2L\beta); \end{aligned}\right\} \tag{ii}$$

and the couples exerted on the constraints at the fixed end will be given by

$$\left.\begin{aligned} L_F &= lT_0+m(M_0+LV_0) = l\frac{C}{L}\tau-2m\frac{B}{L^2}(3\mathrm{w}+L\beta), \\ M_F &= mT_0-l(M_0+LV_0) = m\frac{C}{L}\tau+2l\frac{B}{L^2}(3\mathrm{w}+L\beta). \end{aligned}\right\} \tag{iii}$$

† Now, however, B relates to flexure involving deflexions perpendicular to the (x, y) plane: previously it related to flexure in this plane.

Influence coefficients

57. Substituting for β and τ, we deduce that

$$\left.\begin{aligned}
-Z_F = Z_M &= -6\frac{B}{L^3}(2\mathbf{w}+mL\mathbf{p}-lL\mathbf{q}),\\
L_M &= -\frac{C}{L}(l^2\mathbf{p}+lm\mathbf{q})-2\frac{B}{L^2}\{3m\mathbf{w}+2L(m^2\mathbf{p}-lm\mathbf{q})\},\\
L_F &= \frac{C}{L}(l^2\mathbf{p}+lm\mathbf{q})-2\frac{B}{L^2}\{3m\mathbf{w}+L(m^2\mathbf{p}-lm\mathbf{q})\},\\
M_M &= -\frac{C}{L}(lm\mathbf{p}+m^2\mathbf{q})+2\frac{B}{L^2}\{3l\mathbf{w}+2L(lm\mathbf{p}-l^2\mathbf{q})\},\\
M_F &= \frac{C}{L}(lm\mathbf{p}+m^2\mathbf{q})+2\frac{B}{L^2}\{3l\mathbf{w}+L(lm\mathbf{p}-l^2\mathbf{q})\},
\end{aligned}\right\} \qquad \text{(iv)}$$

whence we have formulae as under for the influence coefficients relevant to grid frameworks:

for the fixed end:

$$\left.\begin{aligned}
&\widehat{zz}_F = 12\frac{B}{L^3}, \quad \widehat{pp}_F = \frac{Cl^2-2Bm^2}{L}, \quad \widehat{qq}_F = \frac{Cm^2-2Bl^2}{L},\\
&-\widehat{pz}_F = \widehat{zp}_F = 6\frac{B}{L^2}m, \quad \widehat{qz}_F = -\widehat{zq}_F = 6\frac{B}{L^2}l,\\
&\widehat{qp}_F = \widehat{pq}_F = \frac{C+2B}{L}lm.
\end{aligned}\right\} \qquad (19)$$

for the end which is moved:

$$\left.\begin{aligned}
&\widehat{zz}_M = -12\frac{B}{L^3}, \quad \widehat{pp}_M = -\frac{Cl^2+4Bm^2}{L}, \quad \widehat{qq}_M = -\frac{Cm^2+4Bl^2}{L},\\
&\widehat{pz}_M = \widehat{zp}_M = -6\frac{B}{L^2}m, \quad \widehat{qz}_M = \widehat{zq}_M = 6\frac{B}{L^2}l,\\
&\widehat{qp}_M = \widehat{pq}_M = -\frac{C-4B}{L}lm.
\end{aligned}\right\} \qquad (20)$$

An example illustrative of the relaxation procedure

58. A simple example will serve to illustrate the application of these formulae. In Fig. 19 a grid of members rigidly connected at A and B and clamped to the walls (which are indicated by shading) is intended to sustain the floor of a room having the shape of a gnomon. We suppose that it is required to find the effect of a

concentrated load W at A, given that the flexural rigidity B and torsional rigidity C have the same values for every section of every member.

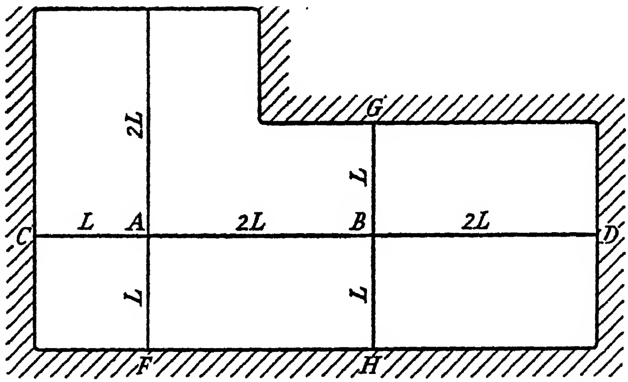

FIG. 19

The operations table

59. Taking Ox, Oy in the directions CD and EF, we have $m = 0$ for the members CA, AB, BD and $l = 0$ for the other members. Therefore, by (19), the effect of a unit displacement w_A imposed at A is to induce forces and couples on the constraints at C, E, B, F as under:

$$\left.\begin{array}{llll} Z_C = 12\dfrac{B}{L^3}, & Z_E = \dfrac{3}{2}\dfrac{B}{L^3}, & Z_B = \dfrac{3}{2}\dfrac{B}{L^3}, & Z_F = 12\dfrac{B}{L^3}, \\ L_C = 0, & L_E = +\dfrac{3}{2}\dfrac{B}{L^2}, & L_B = 0, & L_F = -6\dfrac{B}{L^2}, \\ M_C = -6\dfrac{B}{L^2}, & M_E = 0, & M_B = \dfrac{3}{2}\dfrac{B}{L^2}, & M_F = 0, \end{array}\right\} \quad \text{(i)}$$

by (20), the effect of a unit displacement w_A is to produce actions as under on the constraints at A:†

$$\left.\begin{aligned} Z_A &= -12\frac{B}{L^3}\left(1+\frac{1}{8}+\frac{1}{8}+1\right) = -27\frac{B}{L^3}, \\ L_A &= -6\frac{B}{L^2}\left(0-\frac{1}{4}+0+1\right) = -\frac{9}{2}\frac{B}{L^2}, \\ M_A &= 6\frac{B}{L^2}\left(-1+0+\frac{1}{4}+0\right) = -\frac{9}{2}\frac{B}{L^2}. \end{aligned}\right\} \quad \text{(ii)}$$

† The summations in (ii), (iv) and (v) are a concise representation of calculations made in the manner of § 8, Chap. I.

Dealing in the same way with the effects of a unit rotation p_A imposed at A, we have

$$\left.\begin{aligned}
&Z_C = 0, \quad Z_E = -\frac{3}{2}\frac{B}{L^2}, \quad Z_B = 0, \quad Z_F = +6\frac{B}{L^2},\\
&L_C = \frac{C}{L}, \quad L_E = -\frac{B}{L}, \quad L_B = \frac{1}{2}\frac{C}{L}, \quad L_F = -2\frac{B}{L},\\
&M_C = M_E = M_B = M_F = 0,
\end{aligned}\right\} \quad \text{(iii)}$$

and

$$\left.\begin{aligned}
&Z_A = -6\frac{B}{L^2}\left(0-\frac{1}{4}+0+1\right) = -\frac{9}{2}\frac{B}{L^2},\\
&L_A = -\frac{C}{L}\left(1+0+\frac{1}{2}+0\right)-4\frac{B}{L}\left(0+\frac{1}{2}+0+1\right) = -\frac{3}{2}\frac{C+4B}{L},\\
&M_A = 0.
\end{aligned}\right\} \quad \text{(iv)}$$

Similarly the effects of a unit rotation q_A imposed at A are

$$\left.\begin{aligned}
&Z_C = 6\frac{B}{L^2}, \quad Z_E = 0, \quad Z_B = -\frac{3}{2}\frac{B}{L}, \quad Z_F = 0,\\
&L_C = L_E = L_B = L_F = 0,\\
&M_C = -2\frac{B}{L}, \quad M_E = +\frac{1}{2}\frac{C}{L}, \quad M_B = -\frac{B}{L}, \quad M_F = \frac{C}{L},\\
&Z_A = 6\frac{B}{L^2}\left(-1+0+\frac{1}{4}+0\right) = -\frac{9}{2}\frac{B}{L^2}, \quad L_A = 0,\\
&M_A = -\frac{C}{L}\left(0+\frac{1}{2}+0+1\right)-\frac{4B}{L}\left(1+0+\frac{1}{2}+0\right) = -\frac{3}{2}\frac{C+4B}{L}.
\end{aligned}\right\} \quad \text{(v)}$$

60. These results may be embodied in the first three entries of an Operations Table XIII, and the entries for the other three operations (w_B, p_B, q_B) can be calculated similarly. By expressing C as a definite fraction of B we retain some generality in our treatment, because then the operations required for given effects will all be proportional to $1/B$. We shall assume that

$$C = 0{\cdot}2B,$$

and it will be convenient to divide the moments by L in order to reduce them to pure numbers.

Since it is only at A and B (Fig. 19) that actions have to be liquidated, only the columns relating to these joints will be required in the relaxation process. Table XIII gives these relevant columns,

with operations adjusted in the manner of § 9 so as to make the alteration −1 in each instance as regards the 'corresponding forces'.

TABLE XIII

(Units: any consistent system.)

Operation and multiplier	Z_A	L_A/L	M_A/L	Z_B	L_B/L	M_B/L
1 (*b*) $w_A = \dfrac{L^3}{27B}$	-1	$-0{\cdot}1\dot{6}$	$-0{\cdot}1\dot{6}$	$+0{\cdot}0\dot{5}$	0	$+0{\cdot}0\dot{5}$
2 (*b*) $p_A = \dfrac{L^2}{6{\cdot}3B}$	$-0{\cdot}7143$	-1	0	0	$+0{\cdot}0159$	0
3 (*b*) $q_A = \dfrac{L^2}{6{\cdot}3B}$	$-0{\cdot}7143$	0	-1	$-0{\cdot}2381$	0	$-0{\cdot}1587$
4 (*b*) $w_B = \dfrac{L^3}{27B}$	$+0{\cdot}0\dot{5}$	0	$-0{\cdot}0\dot{5}$	-1	0	0
5 (*b*) $p_B = \dfrac{L^2}{8{\cdot}2B}$	0	$+0{\cdot}0122$	0	0	-1	0
6 (*b*) $q_B = \dfrac{L^2}{4{\cdot}4B}$	$+0{\cdot}34\dot{0}\dot{9}$	0	$-0{\cdot}2\dot{2}\dot{7}$	0	0	-1

61. Again the Relaxation Table is not reproduced in full, but it is summarized in Table XIV for the case in which a unit load is applied at *A* (Fig. 19) and *B* is unloaded. The final displacements have values which may be calculated (when *B* is known) from the multipliers given on the left, and the actions which they impose at *C*, *D*, *E*, *F*, *G*, *H* are shown in the last line of Table XIV.

III. NOTE ON THE EFFECTS OF BENDING IN COMPRESSION MEMBERS

62. We have assumed in this chapter (tacitly) that the actions which come upon a member as a result of distortion occurring in a framework can be combined in accordance with the Principle of Superposition. Actually, if the thrust in a member is appreciable and if the member is appreciably bent, the thrust will induce bending moments (by actions which may be described as '$P.y$ effects')† additional to those which we have considered. In any event bending, if appreciable, will alter the distance between the two ends of a member, and on that account will modify the tension (or thrust) which is entailed by given joint displacements.

† Cf. the notation of §§ 35–9.

Table XIV

Summary of Relaxation Process

	Actions requiring to be liquidated											
	Z_A	L_A/L	M_A/L	Z_B	L_B/L	M_B/L	Z_C	L_C/L	M_C/L	Z_E	L_E/L	M_E/L
Initial action	+1											
$1b$ × +1·4096	−1·4096	−0·2348	−0·2348	+0·0782		+0·0782	+0·6264		−0·3132	+0·0782	+0·0782	
$2b$ × −0·2347	+0·1676	+0·2347			−0·0037			−0·0074		+0·0559	+0·0372	
$3b$ × −0·2701	+0·1929		+0·2701	+0·0643		+0·0429	−0·2572		+0·0857			−0·0043
$4b$ × +0·1424	+0·0079		−0·0079	−0·1424								
$5b$ × −0·0037					+0·0037							
$6b$ × +0·1212	+0·0413		−0·0275			−0·1212						
	+0·0001	−0·0001	−0·0001	+0·0001	0	−0·0001	+0·3692	−0·0074	−0·2275	+0·1341	+0·1154	−0·0043

Table XIV (*continued*)

	Z_F	L_F/L	M_F/L	Z_D	L_D/L	M_D/L	Z_G	L_G/L	M_G/L	Z_H	L_H/L	M_H/L
$1b$ × +1·4096	+0·6264	−0·3132										
$2b$ × −0·2347	−0·2235	+0·0745										
$3b$ × −0·2701			−0·0086									
$4b$ × +0·1424				+0·0079		+0·0079	+0·0632	+0·0316		+0·0632	−0·0316	
$5b$ × −0·0037					0		+0·0027	+0·0009		−0·0027	−0·0009	
$6b$ × +0·1212				−0·0413		−0·0275			+0·0055			+0·0055
	+0·4029	−0·2387	−0·0086	−0·0334	0	−0·0196	+0·0659	+0·0325	+0·0055	+0·0605	−0·0325	+0·0055

Let Ox be drawn through the end joints of a member in their displaced positions, and let the origin O be taken at one end. Let y be the displacement of a section, due to flexure, in a direction perpendicular to this line. Then if s is measured from O along the bent central-line, the distance between the two ends is

$$\int_0^L \frac{dx}{ds}\,ds = \int_0^L \left\{1-\left(\frac{dy}{ds}\right)^2\right\}^{\frac{1}{2}} ds = L-\frac{1}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx$$

very nearly, if dy/dx is everywhere small. So the approach of the two ends on account of bending is measured by

$$\frac{1}{2}\int^L \left(\frac{dy}{dx}\right)^2 dx,$$

and to neutralize this approach we should require to apply an additional tension

$$T = \frac{1}{2}\frac{EA}{L}\int_0^L \left(\frac{dy}{dx}\right)^2 dx. \qquad (21)$$

We have not so far allowed for this effect. The form of (21) shows that it is of the second order, and as such it does not fall within the scope of the Principle of Superposition; but clearly, if at any stage in the relaxation process T is calculated and applied as additional forces on the constraints which control joint displacements, then we can continue the relaxation process so as to liquidate those forces, and in this way it will be possible to arrive, eventually, at a solution in which adequate allowance has been made for '$P.y$ effects'.

63. Accordingly we proceed to examine the form of (21) when y arises from end thrust P acting on a member which is bent by terminal couples M_1, M_2 and by the accompanying shearing actions.

If M_1, M_2 are taken as positive when their sense is such as would make y positive, the equation governing y is

$$-B\frac{d^2y}{dx^2} = Py+M_t, \qquad (22)$$

M_t denoting the bending moment which would exist in the absence of P, so that

$$M_t = M_1+(M_2-M_1)\frac{x}{L}. \qquad (23)$$

Substituting from (23) in (22), when B is uniform and

$$\alpha^2 = \frac{P}{B} \tag{24}$$

we can solve the resulting equation in the form

$$y = \frac{1}{P}\left[\frac{1}{\sin\alpha L}\{M_1\sin\alpha(L-x)+M_2\sin\alpha x\}-M_1-(M_2-M_1)\frac{x}{L}\right], \tag{25}$$

which makes y zero at either end; and then, substituting in (21), we find that

$$T = \frac{1}{2}\frac{EA}{P^2L^2}\left[(M_1^2+M_2^2)\left(\frac{\alpha^2L^2}{2\sin^2\alpha L}+\tfrac{1}{2}\alpha L\cot\alpha L-1\right)-\right.$$
$$\left.-2M_1M_2\left(\frac{\alpha^2L^2}{2\sin^2\alpha L}\cos\alpha L+\frac{\alpha L}{2\sin\alpha L}-1\right)\right]. \tag{26}$$

64. Observing that $B = EAk^2$ where k is the relevant radius of gyration, and substituting for P from (24), we may write (26) in the form

$$\left.\begin{aligned} {}' &= \frac{1}{2B}\frac{L^2}{k^2}[(M_1^2+M_2^2)F_1(\lambda)-2M_1M_2.F_2(\lambda)],\\ \text{where}\qquad \lambda &= \alpha L = L\surd(P/B),\\ F_1(\lambda) &= \frac{\lambda^2+\frac{1}{2}\lambda\sin 2\lambda-1+\cos 2\lambda}{\lambda^4(1-\cos 2\lambda)},\\ F_2(\lambda) &= \frac{\lambda^2\cos\lambda+\lambda\sin\lambda-1+\cos 2\lambda}{\lambda^4(1-\cos 2\lambda)}. \end{aligned}\right\} \tag{27}$$

Hence, given P (and therefore λ), we can calculate the amount T by which bending reduces the thrust in a member when its ends are held at a fixed distance apart. (For this purpose $F_1(\lambda)$ and $F_2(\lambda)$ may be calculated once for all in the range $0 \leqslant \lambda \leqslant \pi$.) We observe that the lowest power of λ in both the numerators and denominator of $F_1(\lambda)$, $F_2(\lambda)$ is λ^6. As $\lambda \to 0$, $F_1(\lambda) \to 1/45$, $F_2(\lambda) \to -7/360$: accordingly

$$T \to \frac{1}{2B}\frac{L^2}{k^2}\frac{1}{180}[4(M_1^2+M_2^2)+7M_1M_2], \tag{28}$$

and this is also the value obtained for T from (21) when we neglect P, in (22), from the beginning.

Having calculated T we must impose this as a tension applied externally to the end joints of the member considered, and then continue the relaxation process. But of course, as a result, the value of the thrust will in general be altered, so strictly speaking our calculated value of T will no longer apply. In the nature of the problem we are obliged to proceed by trial and error.

RECAPITULATION

65. In this chapter we extend to stiff-jointed frameworks the methods which in Chapter I were described in relation to pin-jointed frameworks and in Chapter II were extended to deal with continuous girders. The treatment (which is too detailed to be abstracted with advantage) is based for the most part on the paper cited below as Ref. 3. Further discussion of grid frameworks will be found in a paper by D. G. Christopherson (Ref. 1).

A reader conversant with the Moment Distribution Method of Professor Hardy Cross (Ref. 2) will detect many points of resemblance in this chapter. Some remarks on the interrelation of the two methods will be made in Chapter V.

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IV

THE GENERAL PROBLEM OF SPACE FRAMEWORKS. 'BLOCK' AND 'GROUP RELAXATIONS'

66. THE results of preceding chapters suggest that Relaxation Methods will present no special difficulty (although they may entail more labour) as applied to the general problem of a three-dimensional ('space') framework having joints either 'pinned' or 'stiff'. Given any (self-equilibrating) system of applied forces, we can imagine constraints to operate at every joint whereby the three components of displacement and the three components of rotation can be separately controlled. Then initially, if all constraints are fixed before the forces are applied, all straining of members is prevented and the forces are taken wholly by the constraints;† subsequently, by permitting a suitable relaxation, we can transfer any component of force or moment from a constraint to the framework, and in so doing we shall store strain-energy in the latter. When all joints are 'pinned' and frictionless, the constraints at any one joint have only to control its three component displacements, because no stress is entailed by joint rotations; but if any joint is stiff, it can by rotating transmit couples to adjacent joints through the agency of shearing forces and of the consequent bending moments or (in some instances) by torsional action. The 'unit problem' is accordingly more difficult in the general case: *we shall treat it here in relation to a member of uniform flexural rigidity.*

The unit problem

67. We have to investigate the distortion produced by terminal displacements in a member AB having any arbitrary orientation, but straight and uniform both in cross-section and material. Regarding A as the end which is moved, we take the central line AB as having (initially) the direction Ox', and the principal axes of inertia of the cross-section as having the directions Oy', Oz' respectively, where x', y', z' are related with fixed directions x, y, z by the orthogonal scheme:

† This statement requires qualification if a force is applied otherwise than at a joint; but we have seen (§ 49) that such cases present no difficulty.

	x	y	z
x'	l	m	n
y'	l_1	m_1	n_1
z'	l_2	m_2	n_2

(1)

We suppose that the member is compressed, twisted and bent by forces $\mathbf{X}'$, $\mathbf{Y}'$, $\mathbf{Z}'$ and couples $\mathbf{T}$, $\mathbf{M}_1$, $\mathbf{M}_2$ acting on it at the end (A) which moves: the forces have the directions x', y', z' respectively, and the couples are right-handed with respect to axes having these directions. Then at the section distant r from A the axial thrust is $\mathbf{X}'$, the shearing actions are $\mathbf{Y}'$ and $\mathbf{Z}'$, the twisting couple is $\mathbf{T}$, and the bending moments are

$$\left.\begin{aligned} M_1 &= \mathbf{M}_1 + r\mathbf{Z}', \\ M_2 &= \mathbf{M}_2 - r\mathbf{Y}', \end{aligned}\right\} \tag{2}$$

about axes parallel to Oy', Oz' respectively.

The strain-energies involved by thrust, twist and bending in the planes of the principal axes are (for a straight member) additive. Accordingly (if, as before, we neglect the strain-energy entailed by shear forces) the total strain-energy stored in the member is given by

$$U = \frac{1}{2}\int^{L} \left(\frac{\mathbf{X}'^2}{EA} + \frac{M_1^2}{B_1} + \frac{M_2^2}{B_2} + \frac{\mathbf{T}^2}{C}\right) dr, \tag{3}$$

where (as in preceding chapters)

E denotes Young's modulus for the material,
L is the length and
A is the cross-sectional area of the member,
B_1 $(= EAk_1^2)$ is its flexural rigidity about an axis parallel to Oy',
B_2 $(= EAk_2^2)$ is its flexural rigidity about an axis parallel to Oz', and
C is the torsional rigidity of the cross-section. (4)

Member having both ends stiffly jointed

68. The displacement corresponding with $\mathbf{X}'$ is the component displacement of A in the direction Ox': that is, according to (1), the quantity $(lu_A + mv_A + nw_A)$ where u_A, v_A, w_A are the component

displacements of A in the directions Ox, Oy, Oz. Hence, by Castigliano's first theorem, we have from (3)

$$lu_A+mv_A+nw_A = \frac{\partial U}{\partial \mathbf{X}'} = \int\limits_0^L \frac{\mathbf{X}'}{EA}\,dr = \frac{L}{EA}\mathbf{X}',$$

and in the same way we obtain

$$\left.\begin{aligned} l_1 u_A+m_1 v_A+n_1 w_A &= \frac{\partial U}{\partial \mathbf{Y}'} = \int\limits_0^L \frac{M_2}{B_2}\frac{\partial M_2}{\partial \mathbf{Y}'}\,dr \\ &= -\int\limits_0^L \frac{r}{B_2}(\mathbf{M}_2-r\mathbf{Y}')\,dr \\ &= \frac{L^2}{B_2}(\tfrac{1}{3}L\mathbf{Y}'-\tfrac{1}{2}\mathbf{M}_2), \\ l_2 u_A+m_2 v_A+n_2 w_A &= \frac{\partial U}{\partial \mathbf{Z}'} = \frac{L^2}{B_1}(\tfrac{1}{3}L\mathbf{Z}'+\tfrac{1}{2}\mathbf{M}_1). \end{aligned}\right\} \quad (5)$$

Writing p_A, q_A, r_A for the component rotations of A about axes parallel to Ox, Oy, Oz, we deduce in the same way that

$$\left.\begin{aligned} lp_A+mq_A+nr_A &= \frac{\partial U}{\partial \mathbf{T}} = \frac{L}{C}\mathbf{T}, \\ l_1 p_A+m_1 q_A+n_1 r_A &= \frac{\partial U}{\partial \mathbf{M}_1} = \int\limits_0^L \frac{M_1}{B_1}\frac{\partial M_1}{\partial \mathbf{M}_1}\,dr \\ &= \frac{L}{B_1}(\mathbf{M}_1+\tfrac{1}{2}L\mathbf{Z}'), \\ l_2 p_A+m_2 q_A+n_2 r_A &= \frac{\partial U}{\partial \mathbf{M}_2} = \frac{L}{B_2}(\mathbf{M}_2-\tfrac{1}{2}L\mathbf{Y}'). \end{aligned}\right\} \quad (6)$$

The last two of (5) and the last two of (6) may be replaced by

$$\left.\begin{aligned} \frac{L^3}{B_2}\mathbf{Y}' &= 12(l_1 u_A+m_1 v_A+n_1 w_A)+6L(l_2 p_A+m_2 q_A+n_2 r_A), \\ \frac{L^3}{B_1}\mathbf{Z}' &= 12(l_2 u_A+m_2 v_A+n_2 w_A)-6L(l_1 p_A+m_1 q_A+n_1 r_A), \\ \frac{L^2}{B_1}\mathbf{M}_1 &= -6(l_2 u_A+m_2 v_A+n_2 w_A)+4L(l_1 p_A+m_1 q_A+n_1 r_A), \\ \frac{L^2}{B_2}\mathbf{M}_2 &= 6(l_1 u_A+m_1 v_A+n_1 w_A)+4L(l_2 p_A+m_2 q_A+n_2 r_A). \end{aligned}\right\} \quad (7)$$

Member having one end stiffly, the other freely jointed

69. Evidently, if the joint at A opposes no resistance to a particular rotation, the corresponding moment at A is zero and the corresponding equation (6) is not obtained. If p_A, q_A, r_A can all occur freely, then all of $\mathbf{T}$, $\mathbf{M}_1$, $\mathbf{M}_2$ are zero, and all of equations (6) are suppressed: we are left with the relations

$$\left.\begin{aligned} lu_A+mv_A+nw_A &= \frac{L}{EA}\mathbf{X}', \\ l_1 u_A+m_1 v_A+n_1 w_A &= \frac{1}{3}\frac{L^3}{B_2}\mathbf{Y}', \\ l_2 u_A+m_2 v_A+n_2 w_A &= \frac{1}{3}\frac{L^3}{B_1}\mathbf{Z}', \end{aligned}\right. \tag{5 a}$$

which replace (5), while (6) are replaced by

$$\mathbf{T} = \mathbf{M}_1 = \mathbf{M}_2 = 0. \tag{6 a}$$

The circumstances are different if it is at the other end (B) that no resistance is offered to rotation. Then $\mathbf{T}$ must vanish as before, and $M_1 = M_2 = 0$ when $r = L$; so we have in place of (2)

$$\begin{aligned} M_1 &= (r-L)\mathbf{Z}', \\ M_2 &= (L-r)\mathbf{Y}', \end{aligned} \tag{2 b}$$

and when these substitutions are made in (3) the equations (5) are replaced by

$$\left.\begin{aligned} lu_A+mv_A+nw_A &= \frac{L}{EA}\mathbf{X}' \text{ (as before)}, \\ l_1 u_A+m_1 v_A+n_1 w_A+L(l_2 p_A+m_2 q_A+n_2 r_A) &= \frac{\partial U}{\partial \mathbf{Y}'} \\ &= \int\limits_0^L (L-r)^2\frac{\mathbf{Y}'}{B_2}\,dr = \frac{1}{3}\frac{L^3}{B_2}\mathbf{Y}', \\ l_2 u_A+m_2 v_A+n_2 w_A-L(l_1 p_A+m_1 q_A+n_1 r_A) &= \frac{1}{3}\frac{L^3}{B_1}\mathbf{Z}'. \end{aligned}\right\} \tag{5 b}$$

Equations (5 b) determine $\mathbf{X}'$, $\mathbf{Y}'$, $\mathbf{Z}'$, and then the couples at A are given by

$$\mathbf{T} = 0, \qquad \mathbf{M}_1 = -L\mathbf{Z}', \qquad \mathbf{M}_2 = L\mathbf{Y}'. \tag{6 b}$$

(The first of these equations implies that no resistance is offered to a combined rotation in which $lp_A+mq_A+nr_A = 0$.)

Cases can, of course, occur in which either one, two or three components of rotation are resisted at a particular joint. In Chapter I we dealt with members freely jointed at both ends; we have dealt here with a member attached either to two stiff joints or to one stiff and one free joint; and the procedure required in other cases is obvious.

Influence coefficients

70. Let X, Y, Z be the forces in the directions of x, y, z and M_x, M_y, M_z the couples, right-handed with respect to axes having those directions, which are statically equivalent to $\mathbf{X}'$, $\mathbf{Y}'$, $\mathbf{Z}'$, $\mathbf{T}$, $\mathbf{M}_1$, $\mathbf{M}_2$. Then we have from (I)

$$\left.\begin{aligned} X &= l\mathbf{X}'+l_1\mathbf{Y}'+l_2\mathbf{Z}', \\ Y &= m\mathbf{X}'+m_1\mathbf{Y}'+m_2\mathbf{Z}', \\ Z &= n\mathbf{X}'+n_1\mathbf{Y}'+n_2\mathbf{Z}', \\ M_x &= l\mathbf{T}+l_1\mathbf{M}_1+l_2\mathbf{M}_2, \\ M_y &= m\mathbf{T}+m_1\mathbf{M}_1+m_2\mathbf{M}_2, \\ M_z &= n\mathbf{T}+n_1\mathbf{M}_1+n_2\mathbf{M}_2, \end{aligned}\right\} \tag{8}$$

and on substituting in these expressions from (5), (6) and (7) we find that

$$\left.\begin{aligned} X &= \widehat{xx}\,u_A+\widehat{xy}\,v_A+\widehat{xz}\,w_A+\widehat{xp}\,p_A+\widehat{xq}\,q_A+\widehat{xr}\,r_A, \\ Y &= \widehat{yx}\,u_A+\widehat{yy}\,v_A+\widehat{yz}\,w_A+\widehat{yp}\,p_A+\widehat{yq}\,q_A+\widehat{yr}\,r_A, \\ Z &= \widehat{zx}\,u_A+\widehat{zy}\,v_A+\widehat{zz}\,w_A+\widehat{zp}\,p_A+\widehat{zq}\,q_A+\widehat{zr}\,r_A, \\ M_x &= \widehat{px}\,u_A+\widehat{py}\,v_A+\widehat{pz}\,w_A+\widehat{pp}\,p_A+\widehat{pq}\,q_A+\widehat{pr}\,r_A, \\ M_y &= \widehat{qx}\,u_A+\widehat{qy}\,v_A+\widehat{qz}\,w_A+\widehat{qp}\,p_A+\widehat{qq}\,q_A+\widehat{qr}\,r_A, \\ M_z &= \widehat{rx}\,u_A+\widehat{ry}\,v_A+\widehat{rz}\,w_A+\widehat{rp}\,p_A+\widehat{rq}\,q_A+\widehat{rr}\,r_A, \end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned} \widehat{xx} &= \frac{EA}{L}l^2+\frac{12}{L^3}(B_2 l_1^2+B_1 l_2^2), \\ \widehat{yy} &= \frac{EA}{L}m^2+\frac{12}{L^3}(B_2 m_1^2+B_1 m_2^2), \\ \widehat{zz} &= \frac{EA}{L}n^2+\frac{12}{L^3}(B_2 n_1^2+B_1 n_2^2), \qquad \textit{(cont.)} \end{aligned}\right\} \tag{10}$$

$$
\begin{aligned}
\widehat{zy} = \widehat{yz} &= \frac{EA}{L}mn+\frac{12}{L^3}(B_2 m_1 n_1+B_1 m_2 n_2),\\
\widehat{xz} = \widehat{zx} &= \frac{EA}{L}nl+\frac{12}{L^3}(B_2 n_1 l_1+B_1 n_2 l_2),\\
\widehat{yx} = \widehat{xy} &= \frac{EA}{L}lm+\frac{12}{L^3}(B_2 l_1 m_1+B_1 l_2 m_2),\\
\widehat{pp} &= \frac{C}{L}l^2+\frac{4}{L}(B_1 l_1^2+B_2 l_2^2),\\
\widehat{qq} &= \frac{C}{L}m^2+\frac{4}{L}(B_1 m_1^2+B_2 m_2^2),\\
\widehat{rr} &= \frac{C}{L}n^2+\frac{4}{L}(B_1 n_1^2+B_2 n_2^2),\\
\widehat{rq} = \widehat{qr} &= \frac{C}{L}mn+\frac{4}{L}(B_1 m_1 n_1+B_2 m_2 n_2)\\
\widehat{pr} = \widehat{rp} &= \frac{C}{L}nl+\frac{4}{L}(B_1 n_1 l_1+B_2 n_2 l_2),\\
\widehat{qp} = \widehat{pq} &= \frac{C}{L}lm+\frac{4}{L}(B_1 l_1 m_1+B_2 l_2 m_2),\\
\widehat{px} = \widehat{xp} &= \frac{6}{L^2}(B_2-B_1)l_1 l_2,\\
\widehat{qx} = \widehat{xq} &= \frac{6}{L^2}(B_2 l_1 m_2-B_1 l_2 m_1),\\
\widehat{rx} = \widehat{xr} &= \frac{6}{L^2}(B_2 l_1 n_2-B_1 l_2 n_1),\\
\widehat{py} = \widehat{yp} &= \frac{6}{L^2}(B_2 m_1 l_2-B_1 m_2 l_1),\\
\widehat{qy} = \widehat{yq} &= \frac{6}{L^2}(B_2-B_1)m_1 m_2,\\
\widehat{ry} = \widehat{yr} &= \frac{6}{L^2}(B_2 m_1 n_2-B_1 m_2 n_1),\\
\widehat{pz} = \widehat{zp} &= \frac{6}{L^2}(B_2 n_1 l_2-B_1 n_2 l_1),\\
\widehat{qz} = \widehat{zq} &= \frac{6}{L^2}(B_2 n_1 m_2-B_1 n_2 m_1),\\
\widehat{rz} = \widehat{zr} &= \frac{6}{L^2}(B_2-B_1)n_1 n_2.
\end{aligned}
\tag{10}
$$

71. When p_A, q_A, r_A are unresisted (§ 69), only u_A, v_A, w_A appear in (9), and the expressions (10) are replaced by

$$\left.\begin{array}{l} \widehat{xx} = \dfrac{EA}{L}l^2+\dfrac{3}{L^3}(B_2 l_1^2+B_1 l_2^2) \\ \text{and two similar expressions,} \\ \widehat{zy} = \widehat{yz} = \dfrac{EA}{L}mn+\dfrac{3}{L^3}(B_2 m_1 n_1+B_1 m_2 n_2) \\ \text{and two similar expressions.} \end{array}\right\} \quad (10\text{a})$$

When p_B, q_B, r_B are unresisted, by substituting in (8) from (5 b) and (6 b) we obtain the relations (10 a) again, with

$$\left.\begin{array}{l} \widehat{pp} = \dfrac{3}{L}(B_1 l_1^2+B_2 l_2^2), \\ \text{and two similar expressions,} \\ \widehat{rq} = \widehat{qr} = \dfrac{3}{L}(B_1 m_1 n_1+B_2 m_2 n_2), \\ \text{and two similar expressions,} \\ \widehat{px} = \widehat{xp} = \dfrac{3}{L^2}(B_2-B_1)l_1 l_2, \\ \widehat{qx} = \widehat{xq} = \dfrac{3}{L^2}(B_2 l_1 m_2-B_1 l_2 m_1), \\ \widehat{rx} = \widehat{xr} = \dfrac{3}{L^2}(B_2 l_1 n_2-B_1 l_2 n_1), \\ \text{and six similar expressions.} \end{array}\right\} \quad (10\text{b})$$

72. The foregoing expressions can be simplified when the member has the same flexural rigidity against bending in all planes, so that $B_1 = B_2 = B$ (say). Since the direction-cosines given in (1) constitute an orthogonal scheme, we have

$$\left.\begin{array}{l} l^2+l_1^2+l_2^2 = 1, \quad \text{etc.,} \\ lm+l_1 m_1+l_2 m_2 = 0, \quad \text{etc.,} \\ l_1 m_2-l_2 m_1 = n, \quad \text{etc.,}\dagger \end{array}\right\} \quad (11)$$

† It is assumed here that $\begin{vmatrix} l, & m, & n \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} = +1$. This can always be arranged by a suitable choice of the directions Oy', Oz' (§ 67).

and it follows that in this instance the expressions (10) can be replaced by

$$\left.\begin{aligned}
&\widehat{xx},\ \widehat{yy},\ \widehat{zz} = 12\frac{B}{L^3}+F\times(l^2, m^2, n^2),\\
&\widehat{zy} = \widehat{yz} = Fmn, \quad \widehat{xz} = \widehat{zx} = Fnl, \quad \widehat{yx} = \widehat{xy} = Flm,\\
&\widehat{pp},\ \widehat{qq},\ \widehat{rr} = 4\frac{B}{L}+G\times(l^2, m^2, n^2),\\
&\widehat{rq} = \widehat{qr} = Gmn, \quad \widehat{pr} = \widehat{rp} = Gnl, \quad \widehat{qp} = \widehat{pq} = Glm,\\
&\widehat{px} = \widehat{xp} \quad = \widehat{qy} = \widehat{yq} \quad = \widehat{rz} = \widehat{zr} = 0,\\
&-\widehat{qz} = -\widehat{zq} \quad = \widehat{ry} = \widehat{yr} \quad = 6\frac{B}{L^2}l,\\
&-\widehat{rx} = -\widehat{xr} \quad = \widehat{pz} = \widehat{zp} \quad = 6\frac{B}{L^2}m,\\
&-\widehat{py} = -\widehat{yp} \quad = \widehat{qx} = \widehat{xq} \quad = 6\frac{B}{L^2}n,\\
&\text{where} \quad F = \frac{EA}{L}-12\frac{B}{L^3}, \quad G = \frac{C-4B}{L}.
\end{aligned}\right\} \tag{12}$$

Similarly, the formulae (10a) are replaced by

$$\left.\begin{aligned}
&\widehat{xx},\ \widehat{yy},\ \widehat{zz} = 3\frac{B}{L^3}+F_A\times(l^2, m^2, n^2),\\
&\widehat{zy} = \widehat{yz} = F_A mn, \quad \widehat{xz} = \widehat{zx} = F_A nl, \quad \widehat{yx} = \widehat{xy} = F_A lm,\\
&\text{where} \quad F_A = \frac{EA}{L}-3\frac{B}{L^3},
\end{aligned}\right\} \tag{12a}$$

and the formulae (10b) by

$$\left.\begin{aligned}
&\widehat{pp},\ \widehat{qq},\ \widehat{rr} = 3\frac{B}{L}-3\frac{B}{L}\times(l^2, m^2, n^2),\\
&\widehat{rq} = \widehat{qr} = -3\frac{B}{L}mn, \quad \widehat{pr} = \widehat{rp} = -3\frac{B}{L}nl, \quad \widehat{qp} = \widehat{pq} = -3\frac{B}{L}lm,\\
&\widehat{px} = \widehat{xp} \quad = \widehat{qy} = \widehat{yq} \quad = \widehat{rz} = \widehat{zr} = 0,\\
&-\widehat{qz} = -\widehat{zq} \quad = \widehat{ry} = \widehat{yr} \quad = 3\frac{B}{L^2}l,\\
&-\widehat{rx} = -\widehat{xr} \quad = \widehat{pz} = \widehat{zp} \quad = 3\frac{B}{L^2}m,\\
&-\widehat{py} = -\widehat{yp} \quad = \widehat{qx} = \widehat{xq} \quad = 3\frac{B}{L^2}n.
\end{aligned}\right\} \tag{12b}$$

Forces and moments imposed upon the joints

73. The quantities X, Y, Z, M_x, M_y, M_z, in (9), are the forces and moments which come *upon* the member at its end (A) which moves. It is easy to see that the forces and moments which in consequence are exerted *by* the member on constraints are as follows:

at the joint which is moved in the relaxation:

$$X_M,\ Y_M,\ Z_M,\ (M_x)_M,\ (M_y)_M,\ (M_z)_M = -(X, Y, Z, M_x, M_y, M_z),$$

at the joint which is not moved:

$$\begin{aligned} X_F,\ Y_F,\ Z_F &= X,\ Y,\ Z, \\ (M_x)_F &= M_x+L(nY-mZ), \\ (M_y)_F &= M_y+L(lZ-nX), \\ (M_z)_F &= M_z+L(mX-lY), \end{aligned}$$

as given by (9).

Using these formulae we can calculate directly, from (10), the influence coefficients which must be inserted in (9) to give the forces and moments *on* the fixed and moving joint. We shall give results only for the special case in which $B_1 = B_2 = B$, so that the expressions (10) may be replaced by (12). On that understanding we obtain

$$\left.\begin{aligned} &\widehat{xx}_F,\ \widehat{yy}_F,\ \widehat{zz}_F = 12\frac{B}{L^3}+F\times(l^2, m^2, n^2), \\ &\widehat{zy}_F = \widehat{yz}_F = Fmn, \qquad \widehat{xz}_F = \widehat{zx}_F = Fnl, \qquad \widehat{yx}_F = \widehat{xy}_F = Flm, \\ &\widehat{pp}_F,\ \widehat{qq}_F,\ \widehat{rr}_F = -2\frac{B}{L}+H\times(l^2, m^2, n^2), \\ &\widehat{rq}_F = \widehat{qr}_F = Hmn, \qquad \widehat{pr}_F = \widehat{rp}_F = Hnl, \qquad \widehat{qp}_F = \widehat{pq}_F = Hlm, \\ &\widehat{px}_F = \widehat{xp}_F \qquad = \widehat{qy}_F = \widehat{yq}_F \qquad = \widehat{rz}_F = \widehat{zr}_F = 0, \\ &\widehat{qz}_F = -\widehat{zq}_F \qquad = -\widehat{ry}_F = \widehat{yr}_F \qquad = 6\frac{B}{L^2}l, \\ &\widehat{rx}_F = -\widehat{xr}_F \qquad = -\widehat{pz}_F = \widehat{zp}_F \qquad = 6\frac{B}{L^2}m, \\ &\widehat{py}_F = -\widehat{yp}_F \qquad = -\widehat{qx}_F = \widehat{xq}_F \qquad = 6\frac{B}{L^2}n, \\ &\text{where} \qquad F = \frac{EA}{L}-12\frac{B}{L^3}, \qquad H = \frac{C+2B}{L}, \end{aligned}\right\} \tag{13}$$

as the influence coefficients appropriate to the joint which is fixed, and

$$\left.\begin{aligned}
&\widehat{xx}_M,\ \widehat{yy}_M,\ \widehat{zz}_M = -12\frac{B}{L^3} - F\times(l^2, m^2, n^2),\\
&\widehat{zy}_M = \widehat{yz}_M = -Fmn,\quad \widehat{xz}_M = \widehat{zx}_M = -Fnl,\\
&\qquad\qquad\qquad\qquad\widehat{yx}_M = \widehat{xy}_M = -Flm,\\
&\widehat{pp}_M,\ \widehat{qq}_M,\ \widehat{rr}_M = -4\frac{B}{L} - G\times(l^2, m^2, n^2),\\
&\widehat{rq}_M = \widehat{qr}_M = -Gmn,\quad \widehat{pr}_M = \widehat{rp}_M = -Gnl,\\
&\qquad\qquad\qquad\qquad\widehat{qp}_M = \widehat{pq}_M = -Glm,\\
&\widehat{px}_M = \widehat{xp}_M \quad = \widehat{qy}_M = \widehat{yq}_M \quad = \widehat{rz}_M = \widehat{zr}_M = 0,\\
&\widehat{qz}_M = \widehat{zq}_M \quad = -\widehat{ry}_M = -\widehat{yr}_M \quad = 6\frac{B}{L^2}l,\\
&\widehat{rx}_M = \widehat{xr}_M \quad = -\widehat{pz}_M = -\widehat{zp}_M \quad = 6\frac{B}{L^2}m,\\
&\widehat{py}_M = \widehat{yp}_M \quad = -\widehat{qx}_M = -\widehat{xq}_M \quad = 6\frac{B}{L^2}n,\\
&\text{where}\qquad F = \frac{EA}{L} - 12\frac{B}{L^3},\qquad G = \frac{C-4B}{L},
\end{aligned}\right\} \tag{14}$$

as the influence coefficients appropriate to the end which is moved. Similar calculations are easily made when the member has one joint stiff and the other free,—equations (12) being replaced for this purpose by (12 a) and (12 b) in turn (so that influence coefficients are available for a relaxation permitted at either joint).

74. It should be emphasized that l, m, n in these formulae are the direction-cosines of AB, *the line drawn from the end which is moved to the end which is fixed.* Having regard to this convention, it is easy to show that the expressions for $\widehat{xx}$, $\widehat{xy}$,..., etc. are equivalent, when terms involving B and C are omitted, to those which were derived by other methods in Chapter I. When $n = 0$ (so that the member lies in the (x, y) plane) they reduce to what were obtained in Chapter III.

Since
$$\frac{B}{L^3} = \frac{EAk^2}{L^3} = \frac{k^2}{L^2}\left(\frac{EA}{L}\right),$$
and since k/L is usually small, the value of F will in general be only slightly dependent on the flexural rigidity B.

Using the formulae, we can calculate the effects of any required joint displacement upon any particular member, therefore upon all of the members which come to any particular joint. Then we can construct an operations table, and use this to liquidate specified forces and moments by the relaxation procedure, exactly as before. The calculations are longer (six columns, in general, are now required for every joint that may be moved), but they involve no new principle and accordingly do not call for detailed description. *It is a very important merit of relaxation methods that even as applied to space frameworks they do not call for geometrical thinking by the computer*: this has been done, once for all, in the work which gave the formulae for the influence coefficients.

Example

1. Taking the framework of Example 3, Chapter I, but now assuming that all joints are stiff, calculate A's vertical displacement on the assumption that for every member B (in all planes) $= 20{,}250$, $C = 4{,}050$ ton-inch units.
[*Ans.* 0·048 inch.]

BLOCK AND GROUP RELAXATIONS

75. In every example which has been treated in preceding chapters, liquidation of the applied forces proceeds with satisfactory rapidity, so that once an operations table has been constructed only a reasonable amount of computation is required in order to obtain results of sufficient accuracy for practical purposes. In the language of mathematics, the relaxation process has been found to be convergent. The theoretical basis of its convergence will receive attention in Chapter V; but theoretical convergence is not always the same thing as rapid convergence—as is shown by the uselessness for practical computation of some series expressions for mathematical functions. Before we can regard a method as really practical, we must know not only that it will give a solution *ultimately*, but that it will do so *quickly*; and value thus attaches to any device which will reduce the number of operations in the liquidation process.

Now the operations contemplated in preceding chapters (i.e. 'joint displacements') may give very slow approach to the required configuration when used in relation to an elongated framework such as is indicated in Fig. 20. This is because the resistance offered to a given displacement of a joint near the free end is much greater when, as in the contemplated 'operation', surrounding joints are assumed to be fixed than it is in the actual framework, where the surrounding

joints are free to move in sympathy; consequently a very small displacement will suffice temporarily to 'liquidate' the force at such a joint, in comparison with the displacement which the joint must undergo before the framework as a whole is in equilibrium.

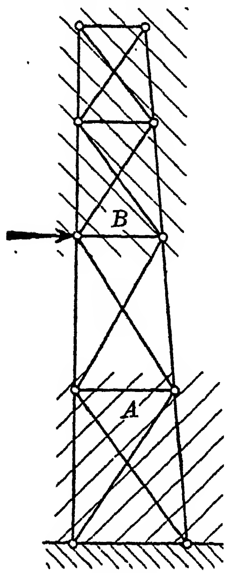

FIG. 20

This difficulty can on the other hand be remedied. If the effects of moving separate joints are known, the principle of superposition can be employed to deduce the effects of any specified combination of joint displacements, and in this way new operations can be added to the existing table, to be used exactly like the simpler operations of joint displacement as a means to liquidation. For example, during an operation in which the parts A and B (indicated by shading in Fig. 20) are moved as rigid bodies either by translation or by rotation, the members which lie wholly within either shaded portion will keep their lengths unaltered, and so will be left unstressed; in general, however, all the members which connect A and B will be strained, because their ends will undergo a relative displacement. A displacement of this kind is termed a **block displacement**. Since the movement of each block is prescribed, the displacement of each end of every member can be calculated; and in order to ascertain the resultant effect of these 'joint displacements' it is only necessary to superpose effects which, taken severally, are already known.

76. We are thus led to consider a new type of operation to which we may give the name **block relaxation**. Whereas hitherto we have contemplated separate constraints of which any one could be relaxed while the others remained fixed, we now imagine constraints whereby particular groups of joints are compelled to move as a whole but allowed, within this restriction, to execute any desired 'rigid-body' displacement. Thus constrained, the group of joints and the members which connect them constitute an inelastic 'block'; and if we imagine the whole framework to be divided into two blocks of this kind, it is clear that a relative displacement can be used to liquidate the *resultant* of the external forces on each. In any relaxation by which a block is moved, positive work will be done by the

external forces and strain-energy will be stored in the framework as before. The resistance to 'block displacement' will come from those members which connect joints of the one block to joints of the other.

The device is not difficult to apply, and in problems concerned with elongated frameworks it permits much more rapid approximation to the required result. This follows from the 'Principle of de St. Venant' (*Elasticity* § 92), which asserts that the effects of any self-equilibrating system of forces are confined (sensibly) to the region in which the system is applied; for the examples of previous chapters have shown that joint displacements will speedily liquidate forces thus localized. Whenever we form two blocks by an arbitrary (and imaginary) 'cut' in the manner of Fig. 20, and then liquidate the resultant action on each, we divide the forces not yet liquidated into two self-equilibrating systems. Every successive 'block relaxation' involves a 'cut' of this kind, and every cut divides the framework into more numerous and smaller blocks, each sustaining external forces which constitute a system in equilibrium.

Looking at the matter in this way, we see that whereas in the standard process of joint relaxation nothing is gained by exact liquidation of a residual force, greater precision may sometimes be desirable in a process of block relaxation.

Forces involved by block displacements in freely jointed frameworks

77. Let A be a joint in the block which is moved, and let it be connected by a member AB with a joint B in the block which is fixed. We shall employ the symbols B_f, B_m to denote the fixed and moving block.

(*a*) '*Block displacement*' $\mathbf{u}$ *in direction* Ox. As in § 8, the forces involved at A and B by a displacement u_A imposed on A are

$$\left.\begin{aligned} -X_A = X_B = u_A \widehat{xx}_{AB}, \\ -Y_A = Y_B = u_A \widehat{yx}_{AB}, \\ -Z_A = Z_B = u_A \widehat{zx}_{AB}. \end{aligned}\right\} \qquad (15)$$

Now let B_m move relatively to B_f through a distance $\mathbf{u}$, as the result of a 'block displacement' confined to the direction Ox. Substituting $\mathbf{u}$ for u_A in (15), we have the forces which come upon B_m and B_f by reason of the member AB; and dealing in the same way with every

member which is affected, we find that the total forces involved by the block displacement are $\mathbf{X}_f$, $\mathbf{Y}_f$, $\mathbf{Z}_f$ on the fixed and $\mathbf{X}_m$, $\mathbf{Y}_m$, $\mathbf{Z}_m$ on the moving block, where

$$\begin{aligned} -\mathbf{X}_m &= \mathbf{X}_f = \mathbf{u}\sum_{f,m}(\widehat{xx}), \\ -\mathbf{Y}_m &= \mathbf{Y}_f = \mathbf{u}\sum_{f,m}(\widehat{yx}), \\ -\mathbf{Z}_m &= \mathbf{Z}_f = \mathbf{u}\sum_{f,m}(\widehat{zx}), \end{aligned} \tag{16}$$

$\sum_{f,m}$ denoting a summation extending to *all members which connect the fixed to the moving block.*

The force imposed on A will have components given by

$$X_A, Y_A, Z_A = -\mathbf{u}\Big[\sum_A{}'(\widehat{xx}),\ \sum_A{}'(\widehat{yx}),\ \sum_A{}'(\widehat{zx})\Big],$$

and the force imposed on B will have components (17)

$$X_B, Y_B, Z_B = \mathbf{u}\Big[\sum_B{}'(\widehat{xx}),\ \sum_B{}'(\widehat{yx}),\ \sum_B{}'(\widehat{zx})\Big],$$

$\sum_A{}'$, $\sum_B{}'$ *denoting summations extending to all members which are attached to A, B and which connect the fixed to the moving block.* The expressions prefixed by a negative sign relate to a joint which is moved, those with a positive sign relate to a joint which is held fixed in the 'block relaxation'.

Having performed the summations involved in (16), we can find a value for the 'block displacement' $\mathbf{u}$ which will 'liquidate' a specified resultant force ($\mathbf{X}$, say) on either block.† Then, having performed the summations involved in (17), we can use those expressions to calculate the forces imposed upon the several joints. The results, as before, can be presented in an operations table: but now, corresponding with a given block displacement, we have to tabulate not only the forces involved at particular joints but also the resultant forces involved on the two blocks; and the relaxation must be specified, not by defining the particular joint which is moved, but by defining the 'cut' whereby the complete framework is divided into 'blocks'.

(*b*) '*Block displacement*' $\mathbf{v}$ *in direction* Oy. Postponing for the moment the question of an appropriate notation for this purpose,

† Since the applied forces constitute an equilibrating system, the resultant forces on B_f and B_m must be equal and opposite.

we remark that the formulae corresponding with (16) and (17) in the case of a 'block displacement' v in the direction Oy are

$$\left.\begin{aligned} -\mathbf{X}_m = \mathbf{X}_f &= \mathrm{v}\sum_{f,m}(\widehat{xy}), \\ -\mathbf{Y}_m = \mathbf{Y}_f &= \mathrm{v}\sum_{f,m}(\widehat{yy}), \\ -\mathbf{Z}_m = \mathbf{Z}_f &= \mathrm{v}\sum_{f,m}(\widehat{zy}), \end{aligned}\right\} \tag{18}$$

and

$$\left.\begin{aligned} X_A, Y_A, Z_A &= -\mathrm{v}\Big[\sum_A{}'(\widehat{xy}),\ \sum_A{}'(\widehat{yy}),\ \sum_A{}'(\widehat{zy})\Big], \\ X_B, Y_B, Z_B &= \mathrm{v}\Big[\sum_B{}'(\widehat{xy}),\ \sum_B{}'(\widehat{yy}),\ \sum_B{}'(\widehat{zy})\Big]. \end{aligned}\right\} \tag{19}$$

(c) *'Block displacement'* w *in direction Oz.* The relevant formulae are

$$\left.\begin{aligned} -\mathbf{X}_m = \mathbf{X}_f &= \mathrm{w}\sum_{f,m}(\widehat{xz}), \\ -\mathbf{Y}_m = \mathbf{Y}_f &= \mathrm{w}\sum_{f,m}(\widehat{yz}), \\ -\mathbf{Z}_m = \mathbf{Z}_f &= \mathrm{w}\sum_{f,m}(\widehat{zz}), \end{aligned}\right\} \tag{20}$$

and

$$\left.\begin{aligned} X_A, Y_A, Z_A &= -\mathrm{w}\Big[\sum_A{}'(\widehat{xz}),\ \sum_A{}'(\widehat{yz}),\ \sum_A{}'(\widehat{zz})\Big], \\ X_B, Y_B, Z_B &= \mathrm{w}\Big[\sum_B{}'(\widehat{xz}),\ \sum_B{}'(\widehat{yz}),\ \sum_B{}'(\widehat{zz})\Big]. \end{aligned}\right\} \tag{21}$$

Forces involved by 'block rotations'

78. Constraints of the type considered in § 76 will permit rigid-body rotations as well as translations, and evidently bending actions will be liquidated most effectively by the imposition of such **block rotations**. Accordingly we now extend the calculations of § 77, again imagining that A is a joint in the block (B_m) which is moved, and that A is connected by a member AB with a joint B in the block (B_f) which is fixed.

(d) *'Block rotation'* $\mathbf{r}$ *about an axis parallel to Oz.* Let x_A, y_A now denote the coordinates of A in relation to the axis about which B_m is assumed to rotate, and let the imposed rotation be $\mathbf{r}$. On account of this rotation the displacements of A are

$$u_A = -y_A.\mathbf{r}, \qquad v_A = x_A.\mathbf{r}, \tag{22}$$

and the forces imposed at A and B by the member AB will be given by

$$\left.\begin{aligned} -X_A &= X_B = u_A \widehat{xx}_{AB} + v_A \widehat{xy}_{AB}, \\ -Y_A &= Y_B = u_A \widehat{yx}_{AB} + v_A \widehat{yy}_{AB}, \\ -Z_A &= Z_B = u_A \widehat{zx}_{AB} + v_A \widehat{zy}_{AB}, \end{aligned}\right\} \tag{23}$$

when the expressions (22) are substituted for u_A, v_A.

The forces imposed at A have a moment about the axis of rotation, given by

$$\begin{aligned} L_A &= x_A Y_A - y_A X_A \\ &= y_A u_A \widehat{xx}_{AB} - x_A v_A \widehat{yy}_{AB} + (-x_A u_A + y_A v_A)\widehat{xy}_{AB}, \\ &\qquad\qquad\qquad\qquad \text{when we substitute from (23),} \\ &= \mathbf{r}[-y_A^2 \widehat{xx}_{AB} - x_A^2 \widehat{yy}_{AB} + 2x_A y_A \widehat{xy}_{AB}], \end{aligned}$$

by (22). So the total moments about the axis of rotation of the forces imposed upon the fixed and moving block are $\mathbf{L}_f$, $\mathbf{L}_m$, where

$$-\mathbf{L}_m = \mathbf{L}_f = \mathbf{r} \sum_{f,m} [y^2 \widehat{xx} + x^2 \widehat{yy} - 2xy\, \widehat{xy}], \tag{24}$$

$\sum_{f,m}$ having the same significance as in § 77. In this formula every member which connects the fixed and moving block is represented by three terms of the kind $y^2\widehat{xx}$, each involving an influence coefficient appropriate to that member (and calculated *ab initio*) and one or two coordinates, relative to the axis of rotation, of the end which moves.

Using (24) we can calculate the magnitude of the block rotation $\mathbf{r}$ which will liquidate any given resultant bending actions on the blocks B_f, B_m; knowing $\mathbf{r}$, we can use (22) to calculate the displacement of every joint in B_m; and from (22) and (23) we have for the forces imposed on A and B as a result of $\mathbf{r}$

$$\left.\begin{aligned} X_A &= \mathbf{r} \sum_A{}' [y\,\widehat{xx} - x\,\widehat{xy}], \\ Y_A &= \mathbf{r} \sum_A{}' [y\,\widehat{yx} - x\,\widehat{yy}], \\ Z_A &= \mathbf{r} \sum_A{}' [y\,\widehat{zx} - x\,\widehat{zy}], \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} X_B &= \mathbf{r} \sum_B{}' [-y\,\widehat{xx} + x\,\widehat{xy}], \\ Y_B &= \mathbf{r} \sum_B{}' [-y\,\widehat{yx} + x\,\widehat{yy}], \\ Z_B &= \mathbf{r} \sum_B{}' [-y\,\widehat{zx} + x\,\widehat{zy}], \end{aligned}\right\} \tag{26}$$

where $\sum_A{}'$, $\sum_B{}'$ have the same significance as in § 77.

These expressions correspond with (17) of § 77. A, in (25), is a typical joint of the block which is moved, and B, in (26), is a typical joint of the block which is fixed. The coordinates x, y associated with the influence coefficients of any member are *the coordinates relative to the axis of rotation of that end of the member which moves.*

(*e*) '*Block rotation*' $\mathbf{p}$ *about an axis parallel to* Ox. An exactly similar argument leads to the expressions

$$-\mathbf{M}_m = \mathbf{M}_f = -\mathbf{p}\sum_{f,m}[z^2\widehat{yy}+y^2\widehat{zz}-2yz\,\widehat{yz}], \tag{27}$$

corresponding with (24), for the moments about the axis of rotation which come upon B_m, B_f as the result of a rotation $\mathbf{p}$; and to the expressions

$$\left.\begin{aligned} X_A &= \mathbf{p}\sum_A{}'[z\,\widehat{xy}-y\,\widehat{xz}],\\ Y_A &= \mathbf{p}\sum_A{}'[z\,\widehat{yy}-y\,\widehat{yz}],\\ Z_A &= \mathbf{p}\sum_A{}'[z\,\widehat{zy}-y\,\widehat{zz}], \end{aligned}\right\} \tag{28}$$

and

$$\left.\begin{aligned} X_B &= \mathbf{p}\sum_B{}'[-z\,\widehat{xy}+y\,\widehat{xz}],\\ Y_B &= \mathbf{p}\sum_B{}'[-z\,\widehat{yy}+y\,\widehat{yz}],\\ Z_B &= \mathbf{p}\sum_B{}'[-z\,\widehat{zy}+y\,\widehat{zz}], \end{aligned}\right\} \tag{29}$$

corresponding with (25) and (26), for the forces transferred to a typical joint of B_m and of B_f respectively.

(*f*) '*Block rotation*' $\mathbf{q}$ *about an axis parallel to* Oy. The expressions

$$-\mathbf{N}_m = \mathbf{N}_f = \mathbf{q}\sum[x^2\widehat{zz}+z^2\widehat{xx}-2zx\,\widehat{zx}] \tag{30}$$

correspond with (24) and (27); and the expressions

$$\left.\begin{aligned} X_A &= \mathbf{q}\sum_A{}'[x\,\widehat{xz}-z\,\widehat{xx}],\\ Y_A &= \mathbf{q}\sum_A{}'[x\,\widehat{yz}-z\,\widehat{yx}],\\ Z_A &= \mathbf{q}\sum_A{}'[x\,\widehat{zz}-z\,\widehat{zx}], \end{aligned}\right\} \tag{31}$$

and

$$\left.\begin{aligned} X_B &= \mathbf{q}\sum_B{}'[-x\,\widehat{xz}+z\,\widehat{xx}],\\ Y_B &= \mathbf{q}\sum_B{}'[-x\,\widehat{yz}+z\,\widehat{yx}],\\ Z_B &= \mathbf{q}\sum_B{}'[-x\,\widehat{zz}+z\,\widehat{zx}], \end{aligned}\right\} \tag{32}$$

correspond with (25), (26) and (28), (29).

Illustrative example

79. To illustrate the use of these formulae we shall discuss the problem shown in Fig. 21. A plane framework of five bays is supported at A and G and loaded at N with a force of 1,000 lb.; the reaction at

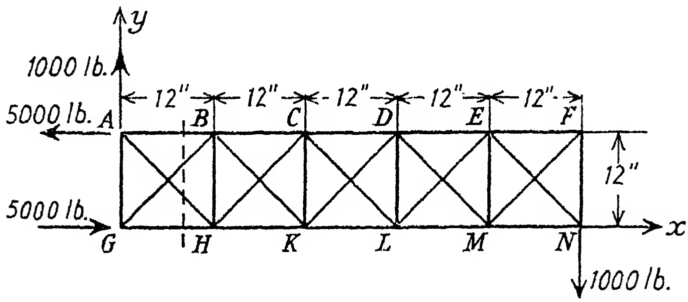

Fig. 21

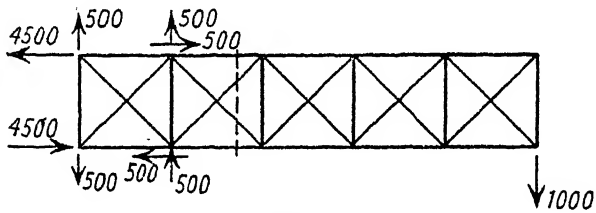

Fig. 22

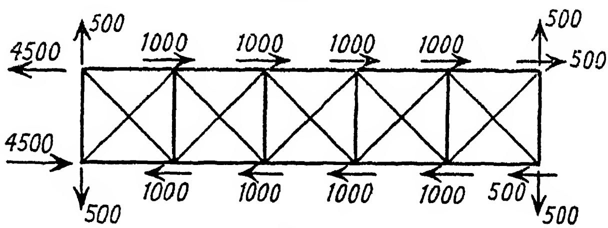

Fig. 23

G is horizontal. All bays will be taken as identical, with elastic properties as under:

$E = 13{,}500$ tons/sq. in. $= 30{,}240{,}000$ lb./sq. in.

Cross-sectional area of top and bottom horizontals, 0·5 sq. in.

Cross-sectional area of diagonals and verticals, 0·25 sq. in.

Table XV corresponds with Table I of Chapter I.

The loading in this problem involves a shearing action, constant throughout the length of the framework, together with a bending action which increases with the distance from the free end. Thus the dominant features of the distortion will be shear strain and flexure.

Table X[illegible]

Units: 1 lb. weight

Member	l	Ω	Δx	Δy	$\Omega.\Delta x$	$\Omega.\Delta y$	$\widehat{xx}$	$\widehat{yy}$	$\widehat{xy}$.
AB, BC, CD, DE, EF, *GH, HK, KL, LM, MN*	1	$1{\cdot}5120\times10^7$	1	0	$1{\cdot}5120\times10^7$	0	$1{\cdot}5120\times10^7$	0	0
AG, BH, CK, DL, EM, FN	1	$0{\cdot}7560\times10^7$	0	1	0	$0{\cdot}7560\times10^7$	0	$0{\cdot}7560\times10^7$	0
AH, BK, CL, DM, EN	1·414	$0{\cdot}2673\times10^7$	1	-1	$0{\cdot}2673\times10^7$	$-0{\cdot}2673\times10^7$	$0{\cdot}2673\times10^7$	$0{\cdot}2673\times10^7$	$-0{\cdot}2673\times10^7$
GB, HC, KD, LE, MF	1·414	$0{\cdot}2673\times10^7$	1	1	$0{\cdot}2673\times10^7$	$0{\cdot}2673\times10^7$	$0{\cdot}2673\times10^7$	$0{\cdot}2673\times10^7$	$0{\cdot}2673\times10^7$

Tab[illegible] X[illegible] I

Units: [illegible] 1 foo[illegible]

'Cut' no.	'Displacement' and joints affected	$-X_m = X_f$	$-Y_m = Y_f$	Forces in direction Ox†	Forces in direction Oy†
1 (*a*)	$\mathbf{v}\,(B, C, D, E, F, H, K, L, M, N) = 1$	*0*	*5,346* $\times 10^3$	$X_A, X_B = -2{,}673\times10^3$ $X_G, X_H = +2{,}673\times10^3$	$Y_A, Y_G = +2{,}673\times10^3$ $Y_B, Y_H = -2{,}673\times10^3$
1 (*b*)	$\mathbf{v}\,(B, C, D, E, F, H, K, L, M, N) = 1{\cdot}87\times10^{-7}$	0	1·00	$X_A, X_B = -0{\cdot}5$ $X_G, X_H = +0{\cdot}5$	$Y_A, Y_G = +0{\cdot}5$ $Y_B, Y_H = -0{\cdot}5$

† Cf. footnote to Table II (p. 10).

We proceed to deal with the former by the method of 'block relaxation', the relevant formulae being (18) and (19).

Imagining a division into blocks by the 'cut' which is indicated by a broken line in Fig. 21, we see that the members which contribute to the summation $\sum_{f,m}$ are AB, GH, AH and GB.† Inserting values from Table XV, we have acccording to (18)

$$
\begin{array}{rll}
-\mathbf{X}_m = \mathbf{X}_f & & \\
= \mathbf{v}\times & 0 & \text{from } AB \\
& 0 & \text{from } GH \\
& -0{\cdot}2673\times 10^7 & \text{from } AH \\
& +0{\cdot}2673\times 10^7 & \text{from } GB \\
\hline
& \text{Sum} = 0 &
\end{array}
\qquad
\begin{array}{rll}
-\mathbf{Y}_m = \mathbf{Y}_f & & \\
= \mathbf{v}\times & 0 & \text{from } AB \\
& 0 & \text{from } GH \\
& 0{\cdot}2673\times 10^7 & \text{from } AH \\
& 0{\cdot}2673\times 10^7 & \text{from } GB \\
\hline
& \text{Sum} = \mathbf{v}\times 5346\times 10^3 &
\end{array}
$$

The joints at which forces are involved are A, B, G, H, and of these B and H share in the displacement $\mathbf{v}$, while A and G are fixed. The members affecting A are AB, AH; so we have according to (19)

$$
X_A = +\mathbf{v}\left|\begin{array}{ll} 0 & \text{from } AB \\ -0{\cdot}2673\times 10^7 & \text{from } AH \end{array}\right.
\qquad
Y_A = +\mathbf{v}\left|\begin{array}{ll} 0 & \text{from } AB \\ +0{\cdot}2673\times 10^7 & \text{from } AH \end{array}\right.
$$
$$
\text{Total} = -2673\times 10^3\,\mathbf{v} \qquad \text{Total} = +2673\times 10^3\,\mathbf{v}
$$

The members affecting G are GB, GH; so we have

$$
X_G = +\mathbf{v}\left|\begin{array}{ll} 0{\cdot}2673\times 10^7 & \text{from } GB \\ 0 & \text{from } GH \end{array}\right.
\qquad
Y_G = +\mathbf{v}\left|\begin{array}{ll} 0{\cdot}2673\times 10^7 & \text{from } GB \\ 0 & \text{from } GH \end{array}\right.
$$
$$
\text{Total} = +2673\times 10^3\,\mathbf{v} \qquad \text{Total} = +2673\times 10^3\,\mathbf{v}
$$

For the joints which are moved in the block displacement we have, similarly,

$$
X_B = -\mathbf{v}\left|\begin{array}{ll} 0 & \text{from } BA \\ 0{\cdot}2673\times 10^7 & \text{from } BG \end{array}\right.
\qquad
Y_B = -\mathbf{v}\left|\begin{array}{ll} 0 & \text{from } BA \\ 0{\cdot}2673\times 10^7 & \text{from } BG \end{array}\right.
$$
$$
\text{Total} = -2673\times 10^3\,\mathbf{v} \qquad \text{Total} = -2673\times 10^3\,\mathbf{v}
$$

$$
X_H = -\mathbf{v}\left|\begin{array}{ll} -0{\cdot}2673\times 10^7 & \text{from } HA \\ 0 & \text{from } HG \end{array}\right.
\qquad
Y_H = -\mathbf{v}\left|\begin{array}{ll} 0{\cdot}2673\times 10^7 & \text{from } HA \\ 0 & \text{from } HG \end{array}\right.
$$
$$
\text{Total} = +2673\times 10^3\,\mathbf{v} \qquad \text{Total} = -2673\times 10^3\,\mathbf{v}
$$

The results of these calculations may be summarized in two lines of an operations table (Table XVI). The second line is derived from the first by multiplying all forces proportionately so as to give an integral value to $\mathbf{Y}_f$.

80. Reverting to Fig. 21, we have to neutralize a resultant force (on the right-hand block) of magnitude 1,000 lb. and having a direction opposite to Oy. Making use for this purpose of the preceding results, we permit a block displacement

$$\mathbf{v} = -1{\cdot}87\times 10^{-4} \text{ (feet)}$$

† These being the members which cross the 'cut'.

on all joints to the right of the cut, and so impose forces (in pounds weight) on the joints A, B, G, H, as under:

$$X_A = +500, \quad X_B = +500, \quad X_G = -500, \quad X_H = -500,$$
$$Y_A = -500, \quad Y_B = +500, \quad Y_G = -500, \quad Y_H = +500.$$

The forces which remain on the constraints after this operation are as shown in Fig. 22.

We now make a new 'cut' to the right of the first, as indicated in Fig. 22 by a broken line, and we neutralize by a further block displacement the resultant shear on the right-hand block. The calculations, being exactly similar to those of § 79, do not call for detailed description. Repeating the process for new cuts made (vertically) through the other bays of the framework, we reduce the unbalanced forces, after five 'block relaxations', to the values given in Fig. 23. Vertical forces have been eliminated, but bending actions are now operative and call for liquidation by means of 'block rotations'.

81. Let a cut be made, as before, in the first bay; let A, G be fixed; and let the block to the right of the cut be given a rotation $\mathbf{r}$ about an axis through the middle point of BH (Fig. 21).

Then in (24)–(26) we have

$$x_B = x_H = 0, \qquad y_B = -y_H = 0{\cdot}5,$$

and on substituting numerical values from Table XV we find that

$$\begin{array}{rll} -\mathbf{L}_m = \mathbf{L}_f = \mathbf{r}\times 0{\cdot}25\times & 1{\cdot}5120\times 10^7 & \text{from } AB \\ & 1{\cdot}5120\times 10^7 & \text{from } GH \\ & 0{\cdot}2673\times 10^7 & \text{from } AH \\ & 0{\cdot}2673\times 10^7 & \text{from } GB \\ \text{Total} = & \underline{\mathbf{r}\times 0{\cdot}25\times 3{\cdot}5586\times 10^7} & \end{array} \tag{33}$$

Similarly from (25) we have, as the forces imposed on the *moved* joints B and H,

$$\begin{array}{rll} X_B = \mathbf{r}\times 0{\cdot}5\times & 1{\cdot}5120\times 10^7 & \text{from } AB \\ & 0{\cdot}2673\times 10^7 & \text{from } GB \\ \text{Total} = & \mathbf{r}\times 0{\cdot}5\times 1{\cdot}7793\times 10^7 & \end{array} \qquad \begin{array}{rll} Y_B = \mathbf{r}\times 0{\cdot}5 & 0 & \text{from } AB \\ & 0{\cdot}2673\times 10^7 & \text{from } GB \\ \text{Total} = & \mathbf{r}\times 0{\cdot}5\times 0{\cdot}2673\times 10^7 & \end{array}$$

and by a similar calculation

$$X_H = -\mathbf{r}\times 0{\cdot}5\times 1{\cdot}7793\times 10^7, \qquad Y_H = \mathbf{r}\times 0{\cdot}5\times 0{\cdot}2673\times 10^7.$$

From (26) we have, as the forces imposed upon the *fixed* joints A and G,

$$\begin{array}{rll} X_A = \mathbf{r}\times 0{\cdot}5 & -1{\cdot}5120\times 10^7 & \text{from } AB \\ & +0{\cdot}2673\times 10^7 & \text{from } AH \\ \text{Total} = & -\mathbf{r}\times 0{\cdot}5\times 1{\cdot}2447\times 10^7 & \end{array} \qquad \begin{array}{rll} Y_A = \mathbf{r}\times 0{\cdot}5 & 0 & \text{from } AB \\ & -0{\cdot}2673\times 10^7 & \text{from } AH \\ \text{Total} = & -\mathbf{r}\times 0{\cdot}5\times 0{\cdot}2673\times 10^7 & \end{array}$$

and by a similar calculation

$$X_G = \mathbf{r}\times 0{\cdot}5\times 1{\cdot}2447\times 10^7, \qquad Y_G = -\mathbf{r}\times 0{\cdot}5\times 0{\cdot}2673\times 10^7.$$

82. The first expression (33) enables us to calculate the value of $\mathbf{r}$ which will liquidate a specified bending action. Changing all of the calculated quantities in proportion, we find that when

then†

$$\left.\begin{aligned} -\mathbf{L}_m = \mathbf{L}_f &= 1, \\ X_B = -X_H &= \frac{2\times 1\cdot 7793}{3\cdot 5586} = 1, \\ -X_A = X_G &= \frac{2\times 1\cdot 2447}{3\cdot 5586} = 0\cdot 700, \\ -Y_A = Y_B = -Y_G = Y_H &= \frac{2\times 0\cdot 2673}{3\cdot 5586} = 0\cdot 150. \end{aligned}\right\} \quad (34)$$

Here we have to liquidate a resultant moment of 4,500 lb.-feet (clockwise) on the right-hand block. We deduce from (33) that

so that

$$\left.\begin{aligned} -0\cdot 25\times 35{,}586\times 10^3\mathbf{r} &= 4{,}500, \\ \mathbf{r} = -\frac{4\cdot 5}{8{,}896\cdot 5} &= -5\cdot 058\times 10^{-4} \text{ (radians)}, \end{aligned}\right\} \quad (35)$$

and from (34), that the forces imposed on A, B, G, H are given by

$$\left.\begin{aligned} X_A = -X_G = 0\cdot 700\times 4{,}500 &= 3{,}150 \text{ lb.}, \\ -X_B = X_H &= 4{,}500 \text{ lb.}, \\ Y_A = -Y_B = Y_G = -Y_H &= 675 \text{ lb.} \end{aligned}\right\} \quad (36)$$

Next, making a cut through the second bay, we liquidate the moment on the right-hand block by imposing a rotation about an axis through the middle point of CK (Fig. 21). The numerical calculations (in this particular example) are the same as before, the resultant moment being now 3,500 lb.-feet, clockwise, on the right-hand block. Equation (35) is replaced by

$$-0\cdot 25\times 35{,}586\times 10^3\mathbf{r} = 3{,}500,$$

whence the required rotation is given by

$$\mathbf{r} = -\frac{3\cdot 5}{8{,}896\cdot 5} = -3\cdot 934\times 10^{-4} \text{ (radians)}. \quad (37)$$

Corresponding with (36) we have

$$\left.\begin{aligned} X_B = -X_H = 0\cdot 700\times 3{,}500 &= 2{,}450 \text{ lb.}, \\ -X_C = X_K &= 3{,}500 \text{ lb.}, \\ Y_B = -Y_C = Y_H = -Y_K &= 525 \text{ lb.} \end{aligned}\right\} \quad (38)$$

† The numbers given in the last two lines of (34) are not exact, but they are correct within 0·0005.

Similar calculations give the 'block rotations' required for liquidating the resultant moments which remain. At the end of this series of operations we are left with unbalanced forces (on the constraints) as shown in Fig. 24.

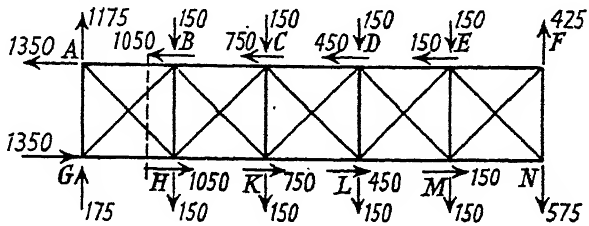

FIG. 24

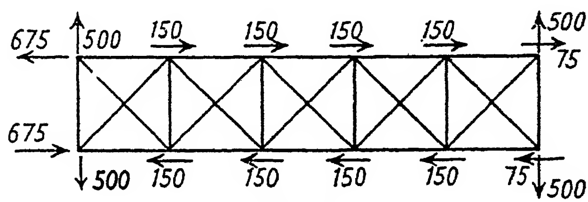

FIG. 25

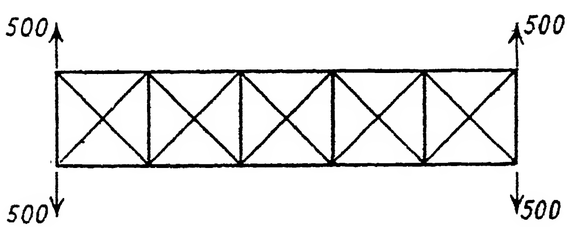

FIG. 26

83. The imposed rotations have eliminated the unbalanced couples, but they have brought new shearing actions into existence. These may now be 'liquidated' by imposing new 'block displacements' in the manner of § 77. Thus the block of four bays which lies to the right of the cut shown by a broken line in Fig. 24 is subjected to a total shearing action of 1,350 lb. in a direction opposite to Oy: this action may be neutralized by permitting a block displacement

$$v = -1{\cdot}35 \times 1{\cdot}87 \times 10^{-4} \text{ (feet)}$$

on all joints to the right of the cut, and so imposing forces (in pounds weight) on the joints A, B, G, H as under:

$$X_A = +675, \quad X_B = +675, \quad X_G = -675, \quad X_H = -675,$$
$$Y_A = -675, \quad Y_B = +675, \quad Y_G = -675, \quad Y_H = +675.$$

When all bays have been thus dealt with, the unbalanced forces which remain are as shown in Fig. 25. Comparing this diagram with Fig. 23, we see that the combined effect of imposing the block rotations of §§ 81–2 and the 'block displacements' of the last paragraph has been to reduce the horizontal forces in the ratio 0·15 : 1 and to leave the vertical forces unchanged: thus the two series of relaxations have served to liquidate 85 per cent. of the horizontal forces shown in Fig. 23, and it follows that if the displacements permitted in each series had been multiplied in the ratio 1/0·85, or $1{\cdot}176_5$, the horizontal forces would have been entirely liquidated, leaving only the forces shown in Fig. 26. These may be treated by the routine methods of joint (as distinct from block) relaxation.

A relatively small change in the data of the problem (viz. partition of the vertical load between joints F and N in equal shares) would have left us at this stage with an exact solution.

Group displacements

84. We remarked in § 75 that the principle of superposition enables us, when the effects of moving any one joint are known, to deduce the effects of any specified combination of joint displacements. So far (§§ 76–83) we have applied this notion only to combinations of a particular kind—namely, 'block displacements'—whereby chosen sets of joints move together as though they constituted a rigid body. But the notion itself is unrestricted: we can combine joint displacements in any way that may be convenient,—the aim (in general) being to reduce the total energy by as great an amount as possible.† We often know intuitively (or by experience) the main features of the distortion which we are seeking to determine, and usually it will be worth while to capitalize this knowledge in the form of new operations added to the operations table. Any such operation (of unrestricted generality) we may term a **group displacement**: 'block displacements' will then be group displacements of a particular kind.

In previous chapters we have made occasional use of group displacements. Thus the operations numbered 6 and 7 in Table II of Chapter I and 7 and 8 in Table X of Chapter III were combinations of joint displacements, chosen for convenience.

† Justification of this rule will be given in Chapter V.

85. It is in deciding on appropriate group displacements that a computer can best make use of his acquired experience: a wise choice will usually simplify the relaxation process, and in some problems it may even make the difference between success and failure. Thus Woods and Warlow-Davies (Ref. 5), seeking to determine the stresses in a typical airship bay (Fig. 27), found that pro-

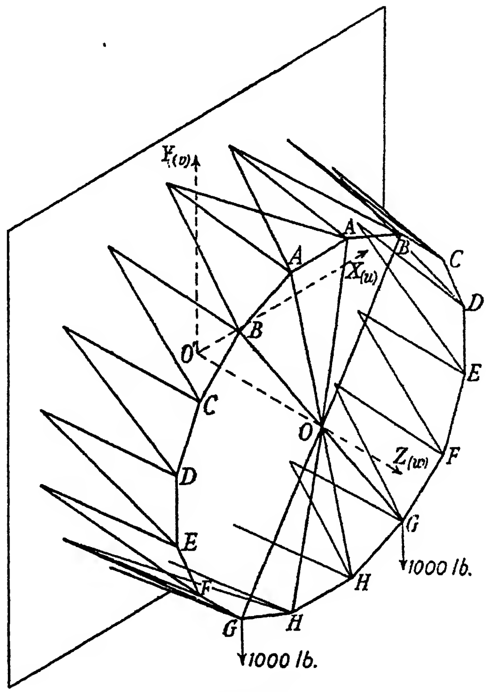

FIG. 27

gress was very slow so long as they restricted their operations to joint displacements, and accordingly made use of group displacements of a kind which they describe as 'quasi-harmonic': that is to say, they allowed all joints of the polygonal ring to make simultaneous displacements such that (e.g.) the radial movement of the jth joint counted from some definite starting-point was given by

$$u_j = A_r \cos rj\alpha, \tag{39}$$

A_r having the same value for all joints, and α denoting the angle subtended at the centre of the polygon by any one of its sides. According to (39) a change made in A_r entails a displacement of every joint, therefore action in every ring member and force on every

constraint of the polygonal ring: these forces Woods and Warlow-Davies analysed into 'quasi-harmonic' components corresponding with their group displacements, thus making possible the construction of an operations table of appropriate kind. For details the reader is referred to their original paper (Ref. 5).

Richards (Ref. 2), dealing with a stiff-jointed framework representative of a steel-framed building ('sky-scraper'), found that torsional actions were liquidated most rapidly by operations of a kind which he termed 'scissor actions'. Fig. 28 shows the nature of these group displacements, each of which involves all joints above some chosen horizontal plane. Christopherson (Ref. 1) emphasizes by an example relating to grid frameworks the importance of a starting approximation which reproduces the main features of the displacement. This may be regarded as a well-chosen group displacement, imposed as the first of a series of 'operations'.

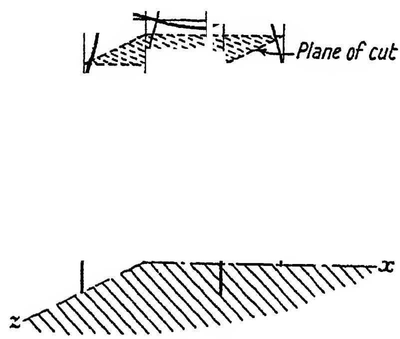

FIG. 28

Optimum magnitude for a group displacement

86. In the nature of the case it is not possible to enunciate principles on which group displacements should be chosen: generally speaking, every example will present its own problem. But assuming its validity we may usefully consider how to satisfy the general rule (§ 84) that the total energy should be reduced as far as is possible.

The *nature* of a group displacement affecting N coordinates will be defined by $(N-1)$ quantities—the ratios of the changes which it imposes; when these have been decided, one more quantity (namely, the change in any one chosen coordinate) will serve to measure the *amount* of the group displacement. By combining joint displacements we can (§ 84) determine the effects of a group displacement on the residual forces when the change in the chosen coordinate is unity:

it will entail calculable changes $\Delta\mathbf{F}_1$, $\Delta\mathbf{F}_2$,..., etc. (for the most part negative), so that (e.g.) the residual force $\mathbf{F}_k$ is altered to $\mathbf{F}_k+\Delta\mathbf{F}_k$. Then from Hooke's law we know that if the change in the chosen coordinate is made x instead of unity, *and if the other coordinates are altered in proportion*, the new values of the residual forces will be $\mathbf{F}_1+x\Delta\mathbf{F}_1$, $\mathbf{F}_2+x\Delta\mathbf{F}_2$,..., etc.

By an operation in which the chosen coordinate is increased by δ the residual force $\mathbf{F}_k$ will be altered to $\mathbf{F}_k+\delta.\Delta\mathbf{F}_k$, and it will do work on its constraint to the amount

$$\int\limits_0^\delta (\mathbf{F}_k+x\Delta\mathbf{F}_k)\,dx = \delta\mathbf{F}_k+\tfrac{1}{2}\delta^2.\Delta\mathbf{F}_k.$$

Similar expressions will hold in respect of the other residual forces, and accordingly the loss of total energy in the operation (i.e. the total work done on all the constraints) will be given by

$$W = \delta\sum_N(\mathbf{F})+\tfrac{1}{2}\delta^2\sum_N(\Delta\mathbf{F}),$$

$\sum_N(\Delta\mathbf{F})$ having a negative value (otherwise W could increase without limit). The only variable in this expression is δ (when the nature of the group displacement is specified), and W regarded as a function of δ will have its maximum value when

$$\sum_N(\mathbf{F})+\delta\sum_N(\Delta\mathbf{F}) = 0.$$

Therefore having decided the nature of a group displacement we can at once calculate the optimum value of its amount: this is given by

$$\delta = -\sum_N(\mathbf{F})\Big/\sum_N(\Delta\mathbf{F}), \tag{40}$$

in which the $\mathbf{F}$'s (initial values) are known and the $\Delta\mathbf{F}$'s are calculable.

RECAPITULATION

87. The first part of this chapter (§§ 67–74) completes (for members of uniform cross-section) the process of generalization which was started in Chapter III. It brings within the scope of the Relaxation Method frameworks having extension in three dimensions, joints either 'pinned' or 'stiff', and members inclined at any angles to the axes of coordinates. The resulting formulae for influence coefficients, *which obviate all need for geometrical thinking*, are strikingly simple

though of necessity numerous (since they have to include so many possibilities). They were first given in the paper cited under Ref. 4.

It is not of course suggested that in practice need will often arise for employing these formulae in their most general form: every engineer knows that an actual structure must be replaced by something simpler if stress-analysis is to yield results that can be translated into improved design. But it has been thought desirable to record the general formulae, leaving simplifying assumptions to be made in relation to particular problems, since thereby means are afforded whereby those assumptions can be put to quantitative test. (For example, displacements can be calculated on the assumption of pinned joints, then inserted in the exact formulae to find values for the 'residual forces' which result. These will indicate clearly the extent to which the assumption was justified.)

The example attached to § 74 (which has been solved without difficulty in a morning)† will indicate to the reader the power of the Relaxation Method, if he will consider what the orthodox alternatives would entail.

88. The remainder of the chapter (based in the main on the paper cited under Ref. 3) deals with ancillary devices ('block' and 'group relaxations') which though dispensable in theory are almost essential to practical success. The reader should give close attention to §§ 75–6 and 84–6, for until he has acquired facility in devising special operations as and when these are required, he can have no true appreciation of the method as an instrument of research.

Facility comes with practice, but experience also has an important part. Although it can be said with literal truth that Relaxation Methods demand no more than a knowledge of arithmetic, nevertheless a 'sense' of structural design has been found to reduce in surprising measure the time required to solve particular problems. Given time, the standard procedure will yield an answer to any example; every operation (properly conducted) is an approach to the desired result. But every major problem, so far, has suggested some new means to more rapid convergence; and each new device is a contribution to the common stock.

† The answer given was obtained (independently) by Mr. D. J. Barclay and Mr. R. W. G. Gandy.

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V

THEORETICAL ASPECTS OF THE RELAXATION METHOD, AND ITS EXTENSION TO OTHER PROBLEMS. THE ADJUSTMENT OF ERRORS

89. It was remarked in Chapter II (§ 30) that as applied to one particular problem (a beam continuous over several rigid supports) the Relaxation Method is identical with the 'Moment Distribution Method' of Professor Hardy Cross. Some account of the latter will conveniently introduce the very important question of convergence. We have still to show that the results of the relaxation process, systematically applied, will in every instance approximate more and more closely to an 'exact' solution.†

90. It will be realized that braced frameworks of the kind considered in Chapter I are merely abstractions, made for mathematical convenience, of frameworks as they occur in practice. The latter are skeletal structures formed by connecting a number of members (generally slender and straight) by joints constructed at their ends. If the joints are suitably designed, the action in every member will be, to a close approximation, simple thrust or tension in a line through the joints which it connects: therefore in the early stages of design we may think of the framework as replaced by a **skeleton diagram** in which the members are replaced by lines of thrust or tension and the joints by **nodal points** in which those lines intersect.‡ Stresses calculated on this understanding (i.e. those which would come upon the members if all joints were free) are termed **primary stresses**. *In most instances they constitute a close approximation* to the true stresses: nevertheless there are always **secondary stresses** which must, if exact results are wanted, be superposed upon them; that is, additional stresses due to flexure resulting from the rigidity of the actual joints.‖

In extreme cases (steel-framed buildings, or trusses of Vierendeel construction) a framework may derive *all* of its stiffness from the rigidity of its joints; then there is (as a rule) no elastic problem of primary stress-determination, and secondary stresses are all-

† That is, to the solution which accords with the data regarded as exact. Cf. Chap. I, §§ 4, 17.

‡ Cf. *Elasticity* § 96.

‖ Ibid., § 108.

important.† It was for problems of this kind that Professor Hardy Cross devised in 1924 (and published in 1929)‡ his 'Moment Distribution Method'. On the ground that the secondary stresses have very little effect on the primary, he argued that for practical purposes the joints may be regarded as fixed in position but free to turn under the action of couples imposed upon them by adjacent members. In consequence the Moment Distribution Method employs the operations which in preceding chapters have been termed 'joint rotations', *but not the operations described as 'joint displacements'*. Otherwise (as applied to frameworks) it is fundamentally identical with the Relaxation Method, though differing in the details of its application.

91. The 'unit problem' in the Moment Distribution Method is the same as what is shown in Fig. 3*a* of Chapter II (§ 24), except that terms in δ are now suppressed because that displacement is prohibited by the underlying assumption. Thus the formulae (iii) of § 24 reduce to

$$\left.\begin{aligned} -Y_M = Y_F &= 6\frac{B}{L^2}r, \\ N_M &= -4\frac{B}{L}r, \\ N_F &= -2\frac{B}{L}r. \end{aligned}\right\} \tag{1}$$

Suppose that we want to liquidate a residual moment N (clockwise) on the constraint at m. The appropriate rotation will be

$$r = \frac{1}{4}\frac{L}{B}\mathrm{N} \tag{2}$$

according to (1); and in consequence a moment given by

$$N_F = -2\frac{B}{L}r = -\tfrac{1}{2}\mathrm{N} \tag{3}$$

(*i.e. counterclockwise*) will be transferred to the constraint at f. Thus the sum of the moments on m and f has been altered by a negative (i.e. counterclockwise) amount $-\tfrac{3}{2}\mathrm{N}$; but statical equilibrium is in fact preserved, because, according to (1), the transferred forces Y_M, Y_F constitute a *clockwise* couple of magnitude

$$6\frac{B}{L}r = \tfrac{3}{2}\mathrm{N}, \quad \text{by (2)}.$$

† Cf. *Elasticity* § 109.

‡ Ref. 4. Valuable accounts of the method have been given by J. F. Baker (Ref. 1) and by H. A. Williams (Ref. 7).

92. These results have an important bearing on the question of the convergence of the Moment Distribution Method. The tendency of Y_M, Y_F is not to produce rotations calling in their turn for liquidation but *joint displacements which are prohibited by the basic assumption of the method*, and for this reason they are usually (and legitimately) left out of account in the distribution process. But in consequence the 'couple account' is thrown out of balance, the liquidated† moment N (§ 91) reappearing as a transferred‡ moment $-\frac{1}{2}$N. It is as though, in the distribution of an estate, death duties were levied at the rate of 150 per cent.

No further argument is required to show that, within the assumptions of the Moment Distribution Method, continued relaxation must inevitably dispose of any system of moments. But the reason is that *moments are in part converted into forces which are neglected*: otherwise a problem would remain. It does remain in relation to braced frameworks, because here the forces predominate in importance, bending moments being small in practice and in theory zero. The Relaxation Method was originally devised for the treatment of braced frameworks, and so from the first its convergency was a fundamental question.

93. That the Relaxation Method does in fact give convergent results can be established by appeal to a general theorem of Mechanics, true whether Hooke's law is satisfied or not: *in any problem of equilibrium we are concerned with a configuration of minimum potential energy.*|| At every step in the relaxation process, if positive work is done on the relaxed constraint, the total energy of the system (i.e. strain-energy stored in the framework *plus* potential energy of the external forces) will be reduced; therefore the system must tend towards the required configuration of equilibrium, in which (by the theorem just stated) this total energy has its minimum value.†† We shall not arrive at the required configuration in any finite number of steps, but we can approach it as closely as we may wish. The approach, as we have seen, can be accelerated by an introduction of 'block relaxations', in which any number of points move together

† In the Moment Distribution Method it is called the moment 'distributed'.

‡ In the Moment Distribution Method it is called the moment 'carried over'.

|| *Elasticity* § 19.

†† Kirchhoff's theorem of uniqueness of solution (*Elasticity* § 14) shows that there is only one configuration of equilibrium, therefore only one absolute minimum of the quantity which is altered in the relaxation process.

like a rigid body, or of 'group relaxations', in which any number move together in some arbitrarily chosen way. (The argument which follows can be extended to such relaxations by a generalization of the notion of 'force' and 'corresponding displacement'.)

94. To fix ideas, suppose that we are concerned with stresses in a pin-jointed framework. According to § 4, the condition of balance for the forces acting in the x-direction on any joint A can be expressed in the form

$$\mathbf{X}_A = \mathsf{X}_A + X_A = 0, \tag{4}$$

X_A denoting the force exerted by the framework and X_A the force applied from outside. The resultant $\mathbf{X}_A$, which must vanish in the required (equilibrium) configuration, is the 'residual force' which makes its appearance in the Relaxation Method.

Now the principle of virtual velocities shows that the force exerted by the joint upon the framework is measured by $\partial\mathfrak{U}/\partial u_A$, $\mathfrak{U}$ denoting the total strain-energy; and evidently this force is equal and opposite to X_A. Also if $\mathfrak{V}$ stands for the potential energy of the external forces, then X_A is measured by $-\partial\mathfrak{V}/\partial u_A$. Therefore equation (4), regarded as an expression for the residual force $\mathbf{X}_A$, may be written in the form

$$\mathbf{X}_A = -\frac{\partial}{\partial u_A}(\mathfrak{U}+\mathfrak{V}), \tag{5}$$

$\mathfrak{U}$ being (when Hooke's law is satisfied) a quadratic function of the displacements u_A, v_A,..., etc., *necessarily positive and with constant coefficients.* $\mathfrak{V}$ is a linear function of the same displacements, also having constant coefficients ($-\mathsf{X}_A$, etc.); so $\mathbf{X}_A$ consists, according to (5), of a constant term and of terms linear in u_A, v_A,..., etc.

95. Differentiating (5) with respect to any one of the displacements, we shall have, for the reason just given, a constant term contributed by $\mathfrak{U}$ and no contribution from $\mathfrak{V}$. Consequently

$$\mathbf{X}_A = \mathsf{X}_A - u_A\frac{\partial^2\mathfrak{U}}{\partial u_A^2} - v_A\frac{\partial^2\mathfrak{U}}{\partial v_A\,\partial u_A} - u_B\frac{\partial^2\mathfrak{U}}{\partial u_B\,\partial u_A} - \ldots, \quad \text{etc.,} \tag{6}$$

and corresponding expressions hold for the other residual forces. Now the entries in the Operations Table which we use in applying the Relaxation Method are figures showing the effects on the residual forces of different joint displacements occurring singly: we

see from (6) that they are in fact second differentials of $\mathfrak{U}$; e.g. the effect on $\mathbf{X}_A$ of a displacement v_k is measured by

$$-v_k \frac{\partial^2 \mathfrak{U}}{\partial v_k \, \partial u_A},$$

and other effects are measured similarly. The identity of the entries in column 1, row 3, and in column 3, row 1 of a typical operations table (e.g. Table II) is thus a necessary consequence of the commutative property

$$\frac{\partial^2}{\partial v_k \, \partial u_A} \equiv \frac{\partial^2}{\partial u_A \, \partial v_k}.$$

96. Expressions of type (6) hold in respect of any elastic system, and according to (6) a residual force $\mathbf{X}_A$ can be brought to zero by imposing a displacement Δu_A, where

$$\Delta u_A = \mathbf{X}_A \Big/ \frac{\partial^2 \mathfrak{U}}{\partial u_A^2}. \tag{7}$$

Now because strain-energy is a necessarily positive quantity, the coefficients of u_A^2, u_B^2,..., etc., in $\mathfrak{U}$ will always be positive, and therefore such differential coefficients as $\partial^2 \mathfrak{U}/\partial u_A^2$. Hence according to (7) *any residual force can be brought to zero by imposing a displacement having the same direction and sense*: this is the basis of the 'relaxation' procedure.

Again, provided that the displacement is not of such magnitude that the sign of the residual force is changed, it will necessarily involve a *decrease* in the value of $(\mathfrak{U}+\mathfrak{V})$, since in that event

$$\delta(\mathfrak{U}+\mathfrak{V}) = -\int \mathbf{X}_A \, du_A \quad \text{according to (5)}$$

will be negative. *The greatest possible decrease for a displacement of given type will be obtained by bringing the corresponding force to zero*: it is $-\Delta(\mathfrak{U}+\mathfrak{V})$, where

$$\begin{aligned} \Delta(\mathfrak{U}+\mathfrak{V}) &= -\tfrac{1}{2} \mathbf{X}_A \Delta u_A \\ &= -\tfrac{1}{2} \mathbf{X}_A^2 \Big/ \frac{\partial^2 \mathfrak{U}}{\partial u_A^2}, \quad \text{according to (7).} \end{aligned}$$

Clearly the best operation to make at any stage is that for which the quantity of type $\mathbf{X}_A^2 \Big/ \frac{\partial^2 \mathfrak{U}}{\partial u_A^2}$ is greatest.

97. The physical argument of § 93 is now confirmed, together with the theorem that in the equilibrium configuration $(\mathfrak{U}+\mathfrak{V})$ has a minimum value—not merely a stationary value, which is all that is stated by equations of the type of (4). For we have shown that the value of $(\mathfrak{U}+\mathfrak{V})$ can be reduced so long as any residual force remains on a constraint, and by a sensible amount so long as any residual force is sensible: therefore relaxation can always be continued until $(\mathfrak{U}+\mathfrak{V})$ has been brought so near to its absolute minimum that all residual forces are negligible.

This physical argument for convergence is due primarily to A. N. Black (Ref. 3). A more rigorous argument (fundamentally similar) has been given by G. Temple in a paper (Ref. 6) which explores the applicability of Relaxation Methods to a very large number of purely mathematical problems. His work receives further notice in Chapter XII.

Application of the Relaxation Method, by analogy, to other physical problems

98. Seen from the standpoint of §§ 93–7 the Relaxation Method is evidently applicable to other physical problems. In a framework we have to determine that distribution of joint displacements which (subject to the overriding condition of continuity) entails a minimum value of the total energy,—this being (by Hooke's law) a quadratic function of the displacements: by analogy, whenever we want the conditions for a minimum value of some quadratic function $\mathfrak{Q}$ of parameters u_A, u_B,..., v_A, v_B,...,† we can treat those parameters as 'displacements' and proceed exactly as before. 'Joint relaxations' will now be changes made, one at a time, in the values of the parameters, and the partial derivatives of $\mathfrak{Q}$ with respect to these parameters will measure 'forces' which the relaxations impose on 'constraints'. Thus every minimal theorem in mathematical physics provides a fresh application of the Relaxation Method.

99. $\mathfrak{U}$ and $\mathfrak{V}$ denoting the quadratic and linear parts of $\mathfrak{Q}$, we may write in conformity with (4) and (5)

$$\mathbf{X}_A = \mathrm{X}_A + X_A = -\frac{\partial}{\partial u_A}(\mathfrak{U}+\mathfrak{V}), \quad ..., \text{etc.},$$

† To preserve the analogy we here divide the parameters into classes; but such grouping is not essential, nor (if it is adopted) is the number of classes in any way restricted.

where $X_A, \ldots$, etc., have values independent of u_A, $v_A, \ldots$, etc. Then $\mathbf{X}_A$ will again be given by the expression (6), whence we have

$$\frac{\partial \mathbf{X}_A}{\partial u_A} = -\frac{\partial^2 \mathfrak{U}}{\partial u_A^2} = -\frac{\partial^2 \mathfrak{Q}}{\partial u_A^2}, \quad \text{since } \mathfrak{P} \text{ is linear in the 'displacements',}$$

$$< 0, \quad \text{by the minimal property of } \mathfrak{Q};$$

and the same argument as before shows that relaxation processes, applied systematically, will give results converging steadily towards an exact solution.†

An exact elastic analogue is obtained if we visualize a framework problem in which $u_A, \ldots$, etc. are joint displacements, $X_A, \ldots$, etc. are externally applied forces, and $\mathbf{X}_A$ as given by (6) is the total force which comes upon the constraint at A. This problem we can treat in the usual way, calculating the effects of joint displacements, and using the device of 'block' and 'group displacements' as may be found convenient.

Adjustment of errors in a level survey

100. A simple example is afforded by the problem of adjusting errors in a survey of levels. Let z_A, $z_B, \ldots$ stand for the heights above datum level of points A, $B, \ldots$ for which observations have been made, and let Δ_{AB} stand for the observed difference of level between A and B (i.e., more precisely, the observed rise in passing from A to B). On account of errors of observation the Δ's will not all be consistent with any one set of values for z_A, $z_B, \ldots$, etc. In other words, if η_{AB} is the 'error' in Δ_{AB}, defined by

$$\eta_{AB} = \Delta_{AB} - (z_B - z_A), \tag{8}$$

some at least of the η's will be finite, whatever values we attach to z_A, $z_B, \ldots$, etc.. According to the theory of errors, our problem is to determine values such that the quantity

$$2\mathfrak{Q} = \sum_m [\omega_{AB}^2 \eta_{AB}^2] \tag{9}$$

(i.e. the sum of the squares of the errors 'weighted' by factors $\omega_{AB}^2, \ldots$, etc. which are given) shall have its minimum value. Initially the z's are unknown, the Δ's are specified.

† i.e. to the conditions for an absolute minimum of $\mathfrak{Q}$. It can usually be proved that $\mathfrak{Q}$ has only one minimum (cf. footnote to § 93).

Substituting for η_{AB} in the expression for $\mathfrak{Q}$, we write in conformity with § 99

$$\mathfrak{Q} = \mathfrak{U}+\mathfrak{V},$$

where

$$\left.\begin{aligned}\mathfrak{U} &= \tfrac{1}{2}\sum_m [\omega_{AB}^2(z_B-z_A)^2],\\ \mathfrak{V} &= \tfrac{1}{2}\sum [\omega_{AB}^2\{\Delta_{AB}^2-2\Delta_{AB}(z_B-z_A)\}],\end{aligned}\right\} \tag{10}$$

and

$$\mathbf{Z}_A = \mathsf{Z}_A+Z_A,$$

where

$$\left.\begin{aligned}\mathsf{Z}_A &= -\frac{\partial\mathfrak{V}}{\partial z_A} = -\sum_A [\omega_{AB}^2\Delta_{AB}],\\ Z_A &= -\frac{\partial\mathfrak{U}}{\partial z_A} = \sum_A [\omega_{AB}^2(z_B-z_A)].\end{aligned}\right\} \tag{11}$$

Here $\sum_m$ denotes a summation extending to every pair of points for which an observation has been made, $\sum_A$ a summation extending to every observation which involves z_A.

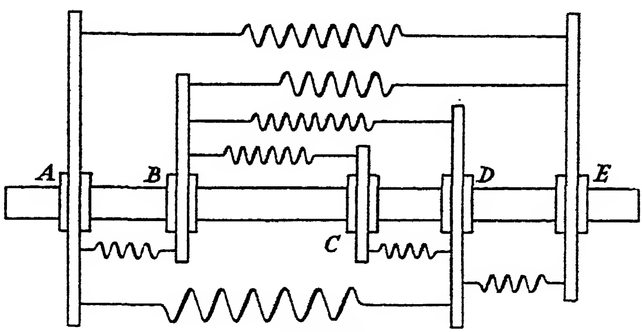

FIG. 29

To obtain an elastic analogue in the manner of § 99 we have only to regard the Z's as external forces and the Z's as forces exerted by the elastic system in virtue of displacements z_A,..., etc. The Z's must vanish with the displacements, which accordingly must be measured in relation to a datum configuration in which no forces are operative. Being all of one kind, the z's are most simply visualized as the displacements of a number of yokes or collars A, B, C, D, E (Fig. 29), which can slide without friction on a straight bar having the direction Oz, and which are connected by springs having elastic properties defined by the ω^2's. The external forces are in equilibrium, since $\sum_j [\mathsf{Z}_A] = 0$, by (11), in virtue of the definition of Δ_{AB}: we assume that initially (when the z's are zero) they are taken wholly by 'constraints', and we transfer them to the springs by relaxations permitted to the collars one at a time.

Alternatively we can construct an elastic analogue which relates to a

self-strained system. Suppose that initially the springs have lengths such that they cannot all be fitted into place without straining, and that η_{AB} is the amount by which the unstrained length (Δ_{AB}) of the spring connecting A and B exceeds the distance $(z_B - z_A)$ between those points. Then by (8), if (as before) ω^2_{AB} defines the elastic properties of this spring, its length can be reduced so as to fit by imposing a compression $\omega^2_{AB}\,\eta_{AB}$: thereby we apply to it at A an external force $\omega^2_{AB}\,\eta_{AB}$ in the z-direction. Dealing in the same way with every other spring, we shall have equilibrium in the datum configuration provided that an external force $\sum_A [\omega^2_{AB}\,\eta_{AB}]$ is applied to A and forces given by similar expressions to the other collars.

Because these forces are not in fact operative, we now have to determine the consequences of neutralizing them by an equal and opposite system of external forces, in which the force at A is $-\sum_A [\omega^2_{AB}\,\eta_{AB}]$. But this is the same problem as before; for initially the force $\mathbf{Z}_A$ on the constraint at A is $-\sum_A [\omega^2_{AB}\,\eta_{AB}]$, and this is the value given by (8) and (11) when all the z's are zero.

101. In practice the labour of computation will be reduced if we start with values for $z_A, z_B,\ldots$, etc., which are nearly correct; and such values are easily obtained, because in a survey the errors of observation are in fact small. For example, if N is the number of points with which we are concerned, and if A is taken as the datum point $(z_A = 0)$, then $(N-1)$ suitably chosen equations of the type

$$\eta_{AB} = \Delta_{AB} - (z_B - z_A) = 0 \tag{12}$$

will serve to fix values for $z_B,\ldots$, etc., and from each of the remaining observations, using (8), we can obtain the (small) value of an error η. According to (8) and (11) we have

$$\mathbf{Z}_A = -\sum_A [\omega^2_{AB}\,\eta_{AB}], \tag{13}$$

and in the summations we shall, proceeding as above, make $(N-1)$ of the η's zero and the remainder small.† The object is to have small forces acting on the constraints initially, instead of the large forces which we should obtain if we started with all the z's zero.

These forces have now to be 'liquidated' by alterations made to the values of $z_A, z_B,\ldots$, etc. Preserving the framework analogy, by systematic relaxation of the constraints we impose 'displacements' $w_A, w_B,\ldots$, etc., on the collars (Fig. 29), and in consequence change

† Which η's are thus made zero is a matter of arbitrary choice, and will not affect the ultimate result. A similar freedom of selection is permitted in problems of self-straining (§ 100).

the residual forces by amounts $\delta Z_A,\ldots$, etc., which can be calculated from (10); for we have as in (6) of § 95

$$\left.\begin{aligned} Z_A &= Z_A - z_A\frac{\partial^2\mathfrak{U}}{\partial z_A^2} - z_B\frac{\partial^2\mathfrak{U}}{\partial z_B\,\partial z_A} - \ldots, \quad \text{etc.}, \\ &\text{and hence (since } w_A = \delta z_A,\ldots, \text{ etc.)} \\ \delta Z_A &= -w_A\frac{\partial\mathfrak{U}}{\partial z_A^2} - w_B\frac{\partial^2\mathfrak{U}}{\partial z_B\,\partial z_A} - \ldots, \quad \text{etc.}, \\ &= -\sum_A [\omega_{AB}^2]w_A + \omega_{AB}^2 w_B + \omega_{AC}^2 w_C + \ldots, \quad \text{etc.} \end{aligned}\right\} \tag{14}$$

We construct a table of standard operations by examining the effects of isolated displacements such as w_A. Thus the changes produced by w_A are:

$$\left.\begin{aligned} &\text{in } Z_B, Z_C, \ldots, \quad && w_A \times (\omega_{AB}^2, \omega_{AC}^2, \ldots), \\ &\text{in } Z_A, && -w_A \sum_A [\omega_{AB}^2], \end{aligned}\right\} \tag{15}$$

and similar expressions hold in respect of other displacements. 'Block' or 'group relaxations' can be derived as required, and since the initial forces on constraints are calculable from (13), we can start a relaxation table and proceed in the usual way.

102. A numerical example will serve to illustrate details. Table XVII records differences of level (i.e. Δ's) as observed in an actual survey of six points N, S, T, R, W, C (the entry $+157{\cdot}26$ in row N, column T, indicating that the point T was 157·26 feet higher than the point N). S was known to be 535·10 feet above datum level.

TABLE XVII. *Values of* Δ

	N	*S*	*T*	*R*	*W*	*C*
N	..	−33·06	+157·26	+164·37	+76·04	+197·28
S	+33·06	..	+190·80	+197·76	+109·16	+230·10
T	−157·26	−190·80	..	+7·22	−81·09	+40·52
R	−164·37	−197·76	−7·22	..	−88·31	+31·80
W	−76·04	−109·16	+81·08	+88·31	..	+121·08
C	−197·28	−230·10	−40·52	−31·80	−121·08	..

The weight (ω^2) to be attached to each observation is given in Table XVIII, and from this a table of standard operations (Table XIX) is constructed according to (15).† (Having assumed that z_S is known, we do not permit a displacement w_S.) It should be observed that Table XVIII, properly interpreted, will itself serve as

† For the system on which Table XIX is formulated, cf. § 9 and Table II.

TABLE XVIII. *Values of* ω^2

	N	S	T	R	W	C
N	..	3·85	2·67	2·99	1·31	1·42
S	3·85	..	2·56	2·93	1·79	2·25
T	2·67	2·56	..	1·49	1·07	1·48
R	2·99	2·93	1·49	..	1·87	1·47
W	1·31	1·79	1·07	1·87	..	2·23
C	1·42	2·25	1·48	1·47	2·23	..

a table of operations: according to (15) we have only to insert in each blank space a figure equal but opposite to the sum of the figures in the same row, and to regard each term of the resulting table as the effect on the 'force' defined by its 'column letter' of a displacement defined by its 'row letter'.†

TABLE XIX

Number and nature of operation		*Effects on the residual forces acting on constraints*					
		$\mathbf{Z}_N$	$\mathbf{Z}_T$	$\mathbf{Z}_R$	$\mathbf{Z}_W$	$\mathbf{Z}_C$	$\mathbf{Z}_S$
1 (*a*)	$w_N = 1$	*−12·24*	*2·67*	*2·99*	*1·31*	*1·42*	*3·85*
(*b*)	$w_N = 81{\cdot}70$	**−1,000**	218	244	107	116	315
2 (*a*)	$w_T = 1$	*2·67*	*−9·27*	*1·49*	*1·07*	*1·48*	*2·56*
(*b*)	$w_T = 107{\cdot}87$	288	**−1,000**	161	115	160	276
3 (*a*)	$w_R = 1$	*2·99*	*1·49*	*−10·75*	*1·87*	*1·47*	*2·93*
(*b*)	$w_R = 93{\cdot}02$	278	139	**−1,000**	174	137	272
4 (*a*)	$w_W = 1$	*1·31*	*1·07*	*1·87*	*−8·27*	*2·23*	*1·79*
(*b*)	$w_W = 120{\cdot}92$	158	129	266	**−1,000**	270	217
5 (*a*)	$w_C = 1$	*1·42*	*1·48*	*1·47*	*2·23*	*−8·85*	*2·25*
(*b*)	$w_C = 112{\cdot}99$	161	167	166	252	**−1,000**	254

103. We now use Table XVII to calculate starting values for the z's and $\mathbf{Z}$'s in the manner of § 101, using the known value $z_S = 535{\cdot}10$, and assuming (for this purpose) that all observations made from S (i.e. all of the bold-type figures in Table XVII) are correct. In this way we obtain the values

$$\left.\begin{array}{lll} z_N = 568{\cdot}16, & z_S = 535{\cdot}10, & z_T = 725{\cdot}90, \\ z_R = 732{\cdot}86, & z_W = 644{\cdot}26, & z_C = 765{\cdot}20, \end{array}\right\} \qquad (16)$$

and from these, taking values of the Δ's and ω^2's from Tables XVII and XVIII, we calculate values of the η's according to (12) and of the $\mathbf{Z}$'s according to (13). The $\mathbf{Z}$'s so found are initial values of the 'forces on constraints': inserting them in the first row of a 'relaxa-

† Thus the figure 2·93 in the fourth row and the second column gives the effect of a displacement z_R on the force $\mathbf{Z}_S$.

tion table', we proceed to 'liquidate' the forces in the usual manner, using 'operations' chosen from Table XIX.

The Relaxation Table (involving 26 operations)† is not reproduced in full but is summarized in the appended Table XX. To obviate decimals all values have been multiplied by 1,000.

TABLE XX. *Summary of Relaxation Process*

Operation and multiplier	Z_N	Z_T	Z_R	Z_W	Z_C	Z_S
1 (*b*) × 1·35	−1,350	294	329	145	157	425
2 (*b*) × −3·44	−991	3,440	−554	−396	−550	−949
3 (*b*) × −0·14	−39	−20	140	−24	−19	−38
4 (*b*) × 0·85	134	110	192	−850	229	185
5 (*b*) × 1·48	238	247	246	373	−1,480	376
(*a*) Totals . .	−2,008	4,071	353	−752	−1,663	−1
(*b*) Initial values .	2,007	−4,074	−348	750	1,665	0
(*c*) Unliquidated remainders .	−1	−3	+5	−2	+2	−1

Stopping the relaxation process at this stage, we deduce (using Table XIX again) the following corrections to the 'datum heights' recorded in (16):

$$\left.\begin{aligned} w_N &= 10^{-3}\times 81{\cdot}70\times 1{\cdot}35 = 0{\cdot}110,\\ w_S &= 0,\\ w_T &= 10^{-3}\times 107{\cdot}87\times(-3{\cdot}44) = -0{\cdot}371,\\ w_R &= 10^{-3}\times 93{\cdot}02\times(-0{\cdot}14) = -0{\cdot}013,\\ w_W &= 10^{-3}\times 120{\cdot}92\times 0{\cdot}85 = 0{\cdot}103,\\ w_C &= 10^{-3}\times 112{\cdot}99\times 1{\cdot}48 = 0{\cdot}167. \end{aligned}\right\} \tag{17}$$

This is our solution, of which the approximation may be judged from the last line of Table XX.

104. The more difficult problem of adjusting errors in a triangulation survey can also be brought within the scope of the method: its 'framework analogue' (§ 99) is rather artificial. For practical details the reader is referred to a paper (Ref. 2) by A. N. Black, by whom this extension of relaxation methods was suggested.

Minimal problems are presented in many branches of mathematical

† The actual sequence of operations was: 2*b* × −4; 3*b* × −1; 5*b* ×0·9; 1*b* ×0·7; 4*b* ×0·4; 3*b* ×0·4; 5*b* ×0·2; 2*b* ×0·2; 1*b* ×0·3; 4*b* ×0·2; 3*b* ×0·2; 5*b* ×0·2; 2*b* ×0·2; 1*b* ×0·2; 4*b* ×0·2; 3*b* ×0·2; 5*b* ×0·2; 2*b* ×0·1; 1*b* ×0·1; 2*b* ×0·05; 3*b* ×0·05; 4*b* ×0·05; 1*b* ×0·05; 5*b* × −0·02; 3*b* ×0·01; 2*b* ×0·01.

physics, and usually the quantity which is to be minimized is a quadratic function of the coordinates. Thus the argument of § 98 has very wide application: in Chapter VI we shall consider some examples which are presented in the theory of electrical networks.

RECAPITULATION

105. This chapter is concerned with fundamental questions,—why, and on what assumptions, the relaxation process is convergent. It shows that as applied to any elastic framework, held in equilibrium by steady forces, the method is in fact convergent because

(i) the equations of equilibrium are conditions for a stationary value of the total energy (*Elasticity* § 19);

(ii) since (as we assume) the equilibrium is *stable*, this stationary value is a *minimum*;

(iii) since the framework is elastic and satisfies Hooke's law, there is only one such minimum value (Kirchhoff's theorem: *Elasticity* § 14), and the equations (i) are linear in respect of the displacements.

In other words, the total energy ($\mathfrak{U}+\mathfrak{V}$) is a quadratic function of the displacements, and its minimum value is wanted. This is the mathematical interpretation of the relaxation process, in which at every stage ($\mathfrak{U}+\mathfrak{V}$) can be decreased by an appropriate 'operation', so long as any force is borne by a 'constraint'. It is due to A. N. Black (Ref. 3), who discovered faults in an earlier attempt to prove convergence (Ref. 5).

106. Because the Relaxation Method was originally devised in relation to *pin-jointed* frameworks (cf. § 1), this question of its convergence was encountered from the beginning, although the first attempt at proof was unsuccessful; and because (presumably) of this focus of attention, its close similarity in principle with the 'Moment Distribution Method' of Professor Hardy Cross was not immediately recognized (Ref. 5, § 37). In that method there is no question of convergence, for reasons (not directly obvious) which are given in §§ 90–2 of this chapter. It did not arise for the reason that the harder structural problem (of stiff-jointed frameworks) engaged Professor Cross's attention from the first.

In so far as the two methods have features in common, priority

is of course held by the Moment Distribution Method.† Evolved in or about the year 1924, this was first brought to public notice in 1929, and at once attracted wide interest. The paper (Ref. 4) is a model of concise exposition.

107. Once the relaxation process is shown to be concerned with the minimum value of a quadratic function, its scope is seen to widen very greatly, since minimal problems are presented in many branches of mathematical physics. A. N. Black (Ref. 2) has developed it in relation to Surveying, and a simple example of such application is given in §§ 100–3. Other applications will be described in Chapter VI.

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† The notion of using methods of successive approximation to calculate stresses in highly redundant structures is of course not new. For the historical note which follows I am indebted to Professor S. P. Timoshenko.

'The first application of the method of successive approximation was given in the book by J. A. L. Waddell, *Bridge Engineering*, 1916. It was applied there in calculating secondary stresses in trusses. A method of successive approximation in analysing Vierendeel trusses was developed by Calisev in his Doctor's thesis, Zagreb. A brief discussion of the same subject was given by him in *Bauingenieur*, 1922, p. 244. . . . More recently two papers have appeared dealing with the history of the methods of successive approximations, namely, by L. C. Maugh, "The analysis of Vierendeel Trusses by successive approximations", and by K. Calisev, "Die Methode der sukzessiven Annäherungen". Both were published in *Publications of the International Association for Bridge and Structural Engineering* (vol. 4 and vol. 5 respectively) and contain references to previous literature.'

VI

RELAXATION METHODS APPLIED TO ELECTRICAL NETWORKS, AND TO 'GYROSTATIC' SYSTEMS. THE 'NORMALIZATION' OF SIMULTANEOUS EQUATIONS

108. In the theory of electrical networks (as presented, for example, in Chapter IX of Ref. 3) some minimal theorems can be enunciated in regard to the heating effects of steady currents. Just as we compare different 'configurations' of an elastic system, so we can compare different 'distributions' of current in the links of a network, and in these theorems such comparison is made between the correct distribution (i.e. the currents which in fact pass under the conditions stated) and other distributions which satisfy *some but not all* of the conditions imposed by 'Kirchhoff's laws'. For proper appreciation of the theorems it is essential that the basis of comparison be clearly stated and understood: that is to say, we must be clear regarding (i) the conditions which are satisfied by *all* of the distributions considered and (ii) the condition which is violated by *all but one*.

Kirchhoff's first law states that there can be no accumulation of electricity at any nodal point of a network. In other words, if I_{BA} stands for the current flowing through any conductor BA towards a nodal point A, then

$$\sum_A [I_{BA}] = 0 \tag{1}$$

when the summation extends to every conductor which is joined with A.

Kirchhoff's second law states that the electric potential of every point in a network must be single-valued. In other words, the total change of potential in any closed circuit must be zero, as calculated from Ohm's law with allowance for the e.m.f.'s of batteries (or alternators) when these are present. For example, in Fig. 30*a* (p. 117) we have (v's standing for potentials and R's for resistances)

$$v_B - v_A = I_{BA} R_{BA}, \quad v_A - v_D = I_{AD} R_{AD}, \quad v_D - v_B = I_{DB} R_{DB} - E,$$

where E is the e.m.f. of the battery shown: therefore according to the second law

$$I_{BA} R_{BA} + I_{AD} R_{AD} + I_{DB} R_{DB} - E = 0.$$

Steady currents

109. Perhaps the most familiar of these minimal theorems is that in which all of the compared distributions satisfy the first of Kirchhoff's laws, but only one satisfies the second. It may be stated thus: *When a system of steady currents flows through a network of conductors containing batteries of which the electromotive forces are specified, the currents are so distributed that the total generation of heat, less twice the output of energy from the batteries, has its minimum value consistent with the satisfaction of Kirchhoff's first law.*

The elastic analogue of this theorem according to § 99 can be shown† to be the second theorem of Castigliano (*Elasticity* § 88) in its most general form,—i.e. as relating to 'self-strained' frameworks. As such it has no obvious connexion with the relaxation process, which is based (cf. Chapter V) on the quite distinct and more general theorem of minimum total energy (*Elasticity* § 19). But a less familiar electrical theorem can be enunciated, of equally wide application, which has an exact analogy with the more general mechanical theorem. In fact, it is most simply demonstrated by appeal to that analogy.

110. This second theorem may be stated as follows: *In a network of conductors to which specified currents are supplied at two or more nodal points,‡ the actual distribution of steady currents is such that the total generation of heat,* less *twice the energy expended in supplying the specified currents from a source at datum potential, has its minimum value consistent with the satisfaction of Kirchhoff's second law.*

To prove the theorem, suppose that two nodal points A and B of a network are connected by a conductor of resistance R_{AB}, and let v_A, v_B denote as before the potentials of A and B. By Ohm's law a current of magnitude $(v_A-v_B)/R_{AB}$ will flow from A to B, so that if I_A, I_B denote the currents flowing towards A and B respectively, then

$$-I_A = I_B = K_{AB}(v_A-v_B),$$

where $K_{AB} = 1/R_{AB}$. Summing as in (1) for all conductors which are connected with A, we have

$$\sum_A [K_{AB}(v_B-v_A)] + \mathsf{I}_A = 0, \tag{2}$$

where I_A stands for the current supplied to A from outside.

† Cf. Ref. 1, § 10.

‡ The algebraic sum of the specified currents must be zero, in order that Kirchhoff's first law may be satisfied. Therefore the datum potential need not be specified: in (3), $\sum_j [\mathsf{I}_A v_0] = v_0 \sum_j [\mathsf{I}_A] = 0$.

Now the heat generated in AB will be measured by $K_{AB}(v_A-v_B)^2$, so the total generation of heat in the network is given by

$$2\mathfrak{U} = \sum_m [K_{AB}(v_A-v_B)^2], \tag{i}$$

$\sum_m$ denoting a summation extending to every conductor. Again, if the current I_A is supplied to A from an outside source at datum potential v_0, the rate at which energy is thereby expended will be measured by $\mathsf{I}_A(v_A-v_0)$, and hence the total energy expended will be measured by

$$\sum_j [\mathsf{I}_A(v_A-v_0)] = -\mathfrak{V} \quad \text{(say)}, \tag{ii}$$

$\sum_j$ denoting a summation extending to every nodal point.

Evidently (2) is typical of the conditions for a minimum value of the quantity

$$\mathfrak{E} = \mathfrak{U}+\mathfrak{V} = \tfrac{1}{2}\sum_m [K_{AB}(v_A-v_B)^2]+\sum_j [\mathsf{I}_A(v_0-v_A)], \tag{3}$$

since it is equivalent to

$$-\frac{\partial\mathfrak{E}}{\partial v_A} = -\frac{\partial}{\partial v_A}(\mathfrak{U}+\mathfrak{V}) = 0.$$

Therefore $\mathfrak{U}$ is the total strain-energy, $\mathfrak{V}$ the total potential energy of the external forces, in the elastic analogue which we obtain in the manner of § 99 by interpreting the v's as displacements, the I's as external forces, and the K's as 'spring constants' or 'influence coefficients'; and corresponding with the theorem of minimum total energy in the elastic analogue, the electrical theorem follows immediately. Conversely, by appeal to that theorem equations of type (2) can be formulated, and they are easily soluble by relaxation methods.

111. When batteries are involved, starting with the assumption that the whole e.m.f. of each battery is utilized in passing current to earth through the resistance of its own associated 'link', we can obtain a 'datum distribution' in which known currents enter and leave the network at nodal points. Then we have merely to calculate and superpose the effects of neutralizing currents supplied at those points; and this modified problem comes within the scope of the theorem stated in § 110.

Example

1. The network shown in Fig. $30a$ is made up of conductors having resistances as under:

$$R_{AB} = 100, \quad R_{AC} = 20, \quad R_{AD} = 50, \quad R_{BC} = 40, \quad R_{BD} = 25,$$

$$R_{CD} = 30, \quad R_{CE} = 30, \quad R_{BE} = 40, \quad R_{DE} = 50 \text{ ohms},$$

and the battery inserted in the link BD has negligible resistance and an e.m.f. of 2 volts. Calculate the current in each link.

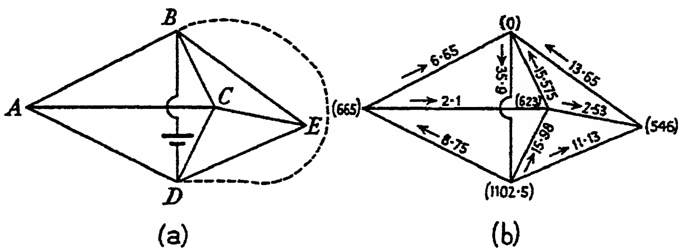

FIG. 30

Here, if B and D were joined by a wire devoid of resistance (as indicated by the broken line in Fig. $30a$), all current passing through the battery from B to D would return by that wire, because the alternative paths offer relatively infinite resistance. Therefore in the datum distribution a current of

$$2/25 = 0{\cdot}08 \text{ amperes}$$

enters the system at B and leaves it at D; and we have now to calculate and superpose the current distribution which results when neutralizing currents $+0{\cdot}08$ and $-0{\cdot}08$ ampere are supplied at D and B to the network *after removal of the battery e.m.f.*

112. The quantity denoted by $\mathfrak{Q}$ in § 110 is now given by

$$2\mathfrak{Q} = \frac{(v_A-v_B)^2}{100}+\frac{(v_A-v_C)^2}{20}+\frac{(v_A-v_D)^2}{50}+\frac{(v_B-v_C)^2}{40}+$$

$$+\frac{(v_B-v_D)^2}{25}+\frac{(v_C-v_D)^2}{30}+\frac{(v_E-v_C)^2}{30}+\frac{(v_E-v_B)^2}{40}+$$

$$+\frac{(v_E-v_D)^2}{50}+2\times 0{\cdot}08[v_0-v_D-(v_0-v_B)], \qquad (4)$$

according to (3); so the 'residual forces' are given by

$$\left.\begin{aligned}
\mathbf{F}_A &= -\frac{\partial \mathfrak{Q}}{\partial v_A} = -\frac{v_A-v_B}{100}-\frac{v_A-v_C}{20}-\frac{v_A-v_D}{50} = 0 \quad \text{initially},\\
\mathbf{F}_B &= -\frac{\partial \mathfrak{Q}}{\partial v_B} = \frac{v_A-v_B}{100}-\frac{v_B-v_C}{40}-\frac{v_B-v_D}{25}-\\
&\qquad -\frac{v_B-v_E}{40}-0{\cdot}08 = -0{\cdot}08 \quad \text{initially},\\
\mathbf{F}_C &= -\frac{\partial \mathfrak{Q}}{\partial v_C} = \frac{v_B-v_C}{40}-\frac{v_C-v_D}{30}+\frac{v_A-v_C}{20}-\\
&\qquad -\frac{v_C-v_E}{30} = 0 \quad \text{initially},\\
\mathbf{F}_D &= -\frac{\partial \mathfrak{Q}}{\partial v_D} = \frac{v_C-v_D}{30}+\frac{v_B-v_D}{25}+\frac{v_A-v_D}{50}-\\
&\qquad -\frac{v_D-v_E}{50}+0{\cdot}08 = 0{\cdot}08 \quad \text{initially},\\
\mathbf{F}_E &= -\frac{\partial \mathfrak{Q}}{\partial v_E} = \frac{v_B-v_E}{40}+\frac{v_C-v_E}{30}+\frac{v_D-v_E}{50} = 0 \quad \text{initially},
\end{aligned}\right\} \tag{5}$$

initial values' signifying values when all of the v's are zero. Hence, examining the effects of increments to v_A, v_B,..., etc. made severally, we deduce an Operations Table as under. (The operations numbered 6 and 7 are simple examples of 'group displacements'.)

TABLE XXI. *Operations for Electrical Network*

(Volt, ampere units.)

No. and nature of operation	$\mathbf{F}_A$	$\mathbf{F}_B$	$\mathbf{F}_C$	$\mathbf{F}_D$	$\mathbf{F}_E$
1 $\Delta v_A =$ 100	-8	1	5	2	0
2 $\Delta v_B =$ 200	2	-20	5	8	5
3 $\Delta v_C =$ 600	30	15	-85	20	20
4 $\Delta v_D =$ 300	6	12	10	-34	6
5 $\Delta v_E =$ 600	0	15	20	12	-47
6 $\Delta v_B =$ 1,000, $\Delta v_D =$ 900	28	-64	55	-62	43
7 $\Delta v_B =$ 600, $\Delta v_D = -300$	0	-72	5	58	9

Inserting the initial forces in the first line of a Relaxation Table (not reproduced), we now proceed to liquidate them in the usual way. Table XXII gives a summary of results after twelve operations (all 'forces' being multiplied by 10^4, so that when all residual forces are reduced below 0·8 they have been liquidated within 0·1 per cent.).

TABLE XXII. *Summary of Relaxation Process*

	F_A	F_B	F_C	F_D	F_E
Initial values . .	0	−800	0	800	0
$v_A \times$ 410 . .	−32·8	4·1	20·5	8·2	0
$v_B \times$ −6,240 . .	−62·4	624	−156	−249·6	−156
$v_C \times$ −12 . .	−0·6	−0·3	1·7	−0·4	−0·4
$v_D \times$ 4,785 . .	95·7	191·4	159·5	−542·3	95·7
$v_E \times$ −780 . .	0	−19·5	−26	−15·6	61·1
Residuals unliquidated .	−0·1	−0·3	−0·3	0·3	0·4

113. According to Table XXII the potentials of A, C, D, E relative to B (after removal of the multiplying factor 10^4: cf. § 112) are approximately

$$0{\cdot}665, \quad 0{\cdot}623, \quad 1{\cdot}102_5 \quad \text{and} \quad 0{\cdot}546 \text{ volts,} \tag{i}$$

and accordingly the currents which flow to B from those points are respectively

$$6{\cdot}65, \quad 15{\cdot}575, \quad 44{\cdot}1 \quad \text{and} \quad 13{\cdot}65 \text{ milliamperes,} \tag{ii}$$

making a total flow to B of 79·975 milliamperes. The currents flowing from D to A, from D to C, and from D to E are

$$\left.\begin{aligned} &\frac{1102{\cdot}5-665}{50} = 8{\cdot}75, \quad \frac{1102{\cdot}5-623}{30} = 15{\cdot}98, \\ \text{and} \quad &\frac{1102{\cdot}5-546}{50} = 11{\cdot}13 \text{ milliamperes, respectively,} \end{aligned}\right\} \tag{iii}$$

making a total flow from D of

$$8{\cdot}75+44{\cdot}1+15{\cdot}98+11{\cdot}13 = 79{\cdot}96 \text{ milliamperes.} \tag{iv}$$

For exact results both totals should have been 80 milliamperes (cf. § 111).

Superposing the current (0·08 amp.) which passed from B to D when these points were joined (§ 111), we find that in the required distribution a current of

$$80-44{\cdot}1 = 35{\cdot}9 \text{ milliamperes} \tag{v}$$

flows from B to D through the resistance (25 ohms) of BD. This entails a potential drop of

$$0{\cdot}0359 \times 25 = 0{\cdot}897_5 \text{ volts,}$$

so that with the battery (of e.m.f. 2 volts) in circuit the p.d. between D and B should be

$$2-0{\cdot}897_5 = 1{\cdot}102_5 \text{ volts,} \tag{vi}$$

agreeing exactly with what was given in (i). The other currents are not altered. The complete solution is exhibited in Fig. 30*b* (p. 117), where bracketed numbers attached to the nodal points give their potentials (relative to v_B) in millivolts, numbered arrows indicate currents in milliamperes.

Alternating currents

114. When alternating instead of direct currents are involved, two quantities instead of one (the resistance) enter into the relation between current and potential difference. Using the symbolic notation which is customary in alternating-current theory, we may say that any link AB of a network has a 'vector admittance'

$$Y_{AB} = g_{AB}+jb_{AB}$$

(the reciprocal of a 'vector impedance' $Z = r-jx$),† and that if

$$V_A = u_A+jv_A, \qquad V_B = u_B+jv_B$$

are the vector potentials of the nodal points A, B, then the vector current flowing from A to B through the link AB is

$$I_{AB} = (g_{AB}+jb_{AB})(V_A-V_B).$$

Hence, if $$I_A = X_A+jY_A = -\sum_A [I_{AB}]$$

is the total vector current flowing *into* A from all links which connect it with other nodal points of the network, then

$$\left.\begin{aligned} -X_A &= \sum_A [g_{AB}(u_A-u_B)-b_{AB}(v_A-v_B)], \\ -Y_A &= \sum_A [b_{AB}(u_A-u_B)+g_{AB}(v_A-v_B)], \end{aligned}\right\} \tag{6}$$

$\sum_A$ having the same significance as in preceding sections. Moreover, if

$$\mathsf{I}_A = \mathsf{X}_A+j\mathsf{Y}_A$$

stands for the vector current supplied to A from outside, then

$$\mathbf{X}_A = \mathsf{X}_A+X_A = 0, \qquad \mathbf{Y}_A = \mathsf{Y}_A+Y_A = 0, \qquad \text{..., etc.} \tag{7}$$

by Kirchhoff's first law, since there can be no accumulation of current at a nodal point.

115. Except in one particular, equations (6) and (7) are exactly analogous with those which govern the joint displacements of a plane pin-jointed framework held in equilibrium by specified forces. But

† The notation of this section is based on that of Steinmetz (Ref. 4, chap. vii), but some new symbols have been introduced with the object of emphasizing the elastic analogue. j stands for $\sqrt{(-1)}$.

in the elastic problem, since X_A, Y_A are partial differentials with respect to u_A, v_A of the same function $\mathfrak{W}$ (the total elastic strain-energy), the 'Maxwell reciprocal relations'

$$\frac{\partial X_A}{\partial v_A} = \frac{\partial Y_A}{\partial u_A} \tag{8}$$

are satisfied (cf. § 95): from (6), on the contrary, we have

$$\frac{\partial X_A}{\partial v_A} = \sum_A [b_{AB}] = -\frac{\partial Y_A}{\partial u_A}, \tag{9}$$

and in this respect the analogy breaks down.

Terms such as those in (6) which involve a factor b_{AB} are sometimes called **gyrostatic** or **non-energetic terms.** Their consequences are important, since they do not permit us to interpret (7) as conditions for a minimal value of some quadratic function of the coordinates. We can interpret them as conditions for a *stationary* value of a certain function (Ref. 1, § 11); *but this function, in the required distribution, though a minimum as regards variations of the u's is a maximum as regards variations of the v's.*

116. In consequence, although the relaxation process is not altered by the circumstance that Maxwell's relations (8) are not satisfied, and in general (suitably applied)† leads quickly to the required solution, it will occasionally fail by giving divergent results, and then a different procedure will be necessary. As an example which yields to the standard treatment we shall calculate the amplitude and phase of the current which passes through the network shown in Fig. 31 when an alternating potential difference is applied at two nodal points. (The analogous problem in relation to a plane framework would be to determine the effects of imposing a relative displacement on two given joints.)

Example

2. An alternating p.d. of 1 volt is applied at the nodal points A, E of the network shown in Fig. 31. The vector admittances of the different links, calculated for the particular frequency of the applied p.d., have values as under:

$$\left.\begin{array}{llll} AB, & 101{\cdot}50+j\,79{\cdot}12 & BD, & 1{\cdot}78+j\,6{\cdot}88 \\ AC, & 7{\cdot}58-j\;\,8{\cdot}99 & BE, & 2{\cdot}36+j\,3{\cdot}72 \\ AD, & 12{\cdot}18-j\;\,2{\cdot}03 & CD, & 2{\cdot}55+j\,0{\cdot}55 \\ BC, & 8{\cdot}73+j\;\,6{\cdot}72 & CE, & 1{\cdot}78+j\,3{\cdot}06 \\ & & DE, & 7{\cdot}26+j\,0{\cdot}73 \end{array}\right\} \tag{10}$$

Calculate the total current which passes through the network.

† At every stage the largest of the 'residual forces' should be liquidated, and by that operation which affects it more than any other.

117. Adopting the procedure suggested in § 111, we first suppose that the vector potential of A is 1 and that B, C, D, E are at zero potential. Then the currents passing from A along the links AB, AC, AD are given by

$$\left.\begin{aligned} I_{AB} = (101{\cdot}5+j\,79{\cdot}12), \qquad I_{AC} = (7{\cdot}58-j\,8{\cdot}99), \\ I_{AD} = (12{\cdot}18-j\,2{\cdot}03), \end{aligned}\right\} \tag{i}$$

and no current will flow in any other link of the circuit. By addition we deduce that (to make the assumed potentials correct) a current

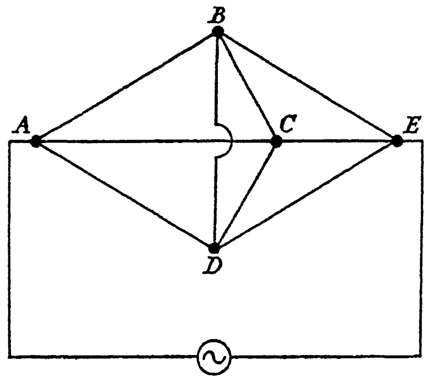

FIG. 31

of amount $(121{\cdot}26+j\,68{\cdot}10)$ will have to be supplied to A from outside. Currents I_{AB}, I_{AC}, I_{AD} will then leave the network at B, C and D respectively.

Actually no current passes to or from the network at B, C or D: therefore we must superpose on the assumed potentials those which would result if currents I_B, I_C, I_D, equal respectively to I_{AB}, I_{AC}, I_{AD} as given in (i), were supplied at B, C, D and allowed to leave the network at A and E, the latter points being held at zero potential. Writing $\mathsf{I}_B = \mathsf{X}_B+j\mathsf{Y}_B$,..., etc., we thus have initially

$$\left.\begin{aligned} \mathbf{X}_A = \mathsf{X}_A = -121{\cdot}26, \quad \mathbf{X}_B = \mathsf{X}_B = 101{\cdot}50, \\ \mathbf{X}_C = \mathsf{X}_C = 7{\cdot}58, \quad \mathbf{X}_D \doteq \mathsf{X}_D = 12{\cdot}18, \\ \mathbf{Y}_A = \mathsf{Y}_A = -68{\cdot}10, \quad \mathbf{Y}_B = \mathsf{Y}_B = 79{\cdot}12, \\ \mathbf{Y}_C = \mathsf{Y}_C = -8{\cdot}99, \quad \mathbf{Y}_D = \mathsf{Y}_D = -2{\cdot}03, \\ \text{with} \qquad \mathbf{X}_E = \mathbf{Y}_E = 0. \end{aligned}\right\} \tag{11}$$

In a framework analogue these would be initial values of forces on constraints, and would have to be liquidated by the imposition of

suitable displacements (actually, vector potentials) at B, C and D, *but not at A and E.*

118. We derive a table of standard operations by calculating the changes in the 'residual forces' (typified by X_A, Y_A) which result from changes made in any one, singly, of the 'displacements' (typified by u_A, v_A). As in a normal elastic problem, since the 'external forces' (typified by X_A, Y_A) are specified and therefore invariant, the partial differentials $\partial\mathrm{X}_B/\partial u_A$, $\partial\mathrm{Y}_B/\partial u_A$,... are identical with $\partial X_B/\partial u_A$, $\partial Y_B/\partial u_A$,..., etc. According to (6) they will be zero unless B is directly linked with A in the electrical network, and when B is so linked they will have constant values as under:

$$\left.\begin{aligned} &\frac{\partial \mathrm{X}_B}{\partial u_A} = g_{AB} = \frac{\partial \mathrm{Y}_B}{\partial v_A}, \quad -\frac{\partial \mathrm{X}_B}{\partial v_A} = b_{AB} = \frac{\partial \mathrm{Y}_B}{\partial u_A}. \\ \text{Similarly,}\quad &\frac{\partial \mathrm{X}_A}{\partial u_A} = \frac{\partial \mathrm{Y}_A}{\partial v_A} = -\sum_A (g_{AB}) = g_{AA} \quad \text{(say)}, \\ -&\frac{\partial \mathrm{X}_A}{\partial v_A} = \frac{\partial \mathrm{Y}_A}{\partial u_A} = -\sum_A (b_{AB}) = b_{AA} \quad \text{(say)}. \end{aligned}\right\} \quad (12)$$

Here $\sum_A$ stands (as before) for a summation extending to every nodal point which is directly linked with A. Each differential gives the effect on some residual force of some particular unit displacement, and we know that a displacement of different magnitude will have a proportionate effect. Table XXIII records the effects of unit operations as deduced from (12) for the example now under discussion. The operations numbered 7 and 8 are simple examples of 'group relaxations'.

119. Starting a relaxation table by recording in the first line initial values of X_A, Y_A,..., etc., as given in (11), we now liquidate these forces in the usual way. The actual table is not reproduced, but it is summarized in Table XXIV.† (The progress of the approximation can be estimated by the extent to which X_A, $-\mathrm{X}_E$ and Y_A, $-\mathrm{Y}_E$ had approached equality. By thirteen operations the values of $(\mathrm{X}_A+\mathrm{X}_E)/(\mathrm{X}_A-\mathrm{X}_E)$ and of $(\mathrm{Y}_A+\mathrm{Y}_E)/(\mathrm{Y}_A-\mathrm{Y}_E)$ were both brought below 1 per cent.)

From the operations and multipliers given in the first column we

† The relaxation was effected by Mr. R. J. Atkinson.

TABLE XXIII†

Column number		1	2	3			4	5	6	
Number and nature of operation	X_A	X_B	X_C	X_D	X_E	Y_A	Y_B	Y_C	Y_D	Y_E
(1) $u_B = 1$	101·50	**−114·37**	8·73	1·78	2·36	79·12	−96·44	6·72	6·88	3·72
(2) $u_C = 1$	7·58	8·73	**−20·64**	2·55	1·78	−8·99	6·72	−1·34	0·55	3·06
(3) $u_D = 1$	12·18	1·78	2·55	**−23·77**	7·26	−2·03	6·88	0·55	−6·13	0·73
(4) $v_B = 1$	−79·12	96·44	−6·72	−6·88	−3·72	101·50	**−114·37**	8·73	1·78	2·36
(5) $v_C = 1$	8·99	−6·72	1·34	−0·55	−3·06	7·58	8·73	**−20·64**	2·55	1·78
(6) $v_D = 1$	2·03	−6·88	−0·55	6·13	−0·73	12·18	1·78	2·55	**−23·77**	7·26
(7) $u_B = 1;\ v_B = 1$	22·38	−17·93	2·01	−5·10	−1·36	180·62	**−210·81**	15·45	8·66	6·08
(8) $u_B = 1;\ v_B = -1$	180·62	**−210·81**	15·45	8·66	6·08	−22·38	17·93	−2·01	5·10	1·36

TABLE XXIV†

Column number		1	2	3			4	5	6	
Operation and multiplier	X_A	X_B	X_C	X_D	X_E	Y_A	Y_B	Y_C	Y_D	Y_E
Initial values	−121·26	101·50	7·58	12·18	0	−68·10	79·12	−8·99	−2·03	0
(1) ×0·9482	96·24	−108·45	8·28	1·69	2·24	75·02	−91·44	6·37	6·52	3·53
(2) ×0·8503	6·45	−2·79	−17·55	2·17	1·51	−7·64	5·71	−1·14	0·47	2·60
(3) ×0·6904	8·41	1·23	1·76	−16·41	5·01	−1·40	4·75	0·38	−4·23	0·50
(4) ×(−0·0289)	2·29	−2·79	0·19	0·20	0·11	−2·93	3·30	−0·25	−0·05	−0·07
(5) ×(−0·1744)	−1·57	1·17	−0·23	0·10	0·53	−1·32	−1·52	3·60	−0·44	−0·32
(6) ×0·0109	0·02	−0·07	−0·01	0·07	−0·01	0·13	0·02	0·03	−0·26	0·08
Final values	−9·42	0·01	0·02	0	9·39	−6·24	−0·05	0	−0·02	6·33

† N.B.—Italic capitals distinguish 'forces' which do not require to be liquidated.

can deduce the vector voltages. Rejecting the fourth decimal place as not reliable, we have

$$\left.\begin{aligned} v_B &= 0{\cdot}948-j\,0{\cdot}029, \\ v_C &= 0{\cdot}850-j\,0{\cdot}174, \\ v_D &= 0{\cdot}690+j\,0{\cdot}011, \end{aligned}\right\} \tag{13}$$

and from the last line of Table XXIV we see that the total current flowing through the network from A to E is

$$9{\cdot}4+j\,6{\cdot}3. \tag{14}$$

A modification of the standard method

120. When as in this example the residual forces decrease with satisfactory rapidity, we may regard the standard procedure as justified by its results; for they must vanish in the exact solution, which is unique.† But examples can be constructed in which, as relaxation proceeds, the residual forces are found to *increase*; and for use in such cases we now describe a modification of the standard procedure which, like that procedure as applied to normal elastic problems, can be shown to lead always to convergent results.

The required configuration is one in which, for each of a number of specified nodal points,

$$\mathbf{X} = \mathbf{Y} = 0, \tag{i}$$

$\mathbf{X}$, $\mathbf{Y}$ being defined by (6) and (7) for a typical point A. If now $\sum_{(J)}$ stands for a summation extending to those nodal points, *but to those alone*, for which (i) is to be satisfied, and if

$$2\mathfrak{W} = \sum_{(J)} [\mathbf{X}^2+\mathbf{Y}^2], \tag{15}$$

then $\mathfrak{W}$ will vanish and so (being necessarily positive) will attain its minimum value in the required configuration. Like the total potential energy in a problem of elastic equilibrium, *it can always be reduced so long as any 'residual force'* ($\mathbf{X}$ *or* $\mathbf{Y}$) *remains unliquidated.*

We can thus treat $\mathfrak{W}$ exactly as in our first example (§ 111) we treated the function denoted by $\mathfrak{E}$. Corresponding with the residual forces of that section we have 'quasi-forces' $\mathbf{X}_A$, $\mathbf{Y}_A$, etc., given in terms of $\mathfrak{W}$ by expressions of the types

$$\mathbf{X}_A = -\frac{\partial \mathfrak{W}}{\partial u_A}, \qquad \mathbf{Y}_A = -\frac{\partial \mathfrak{W}}{\partial v_A}, \tag{16}$$

† We have, in effect, to solve a system of simultaneous linear equations, equal in number to the number of the unknowns.

and corresponding with the influence coefficients derived from $\mathfrak{Q}$ we have 'quasi-influence coefficients' given by such quantities as

$$\frac{\partial^2 \mathfrak{W}}{\partial u_A^2} = -\frac{\partial \mathbf{X}_A}{\partial u_A}, \quad \frac{\partial^2 \mathfrak{W}}{\partial u_A\, \partial v_A} = -\frac{\partial \mathbf{X}_A}{\partial v_A} = -\frac{\partial \mathbf{Y}_A}{\partial u_A}, \quad \frac{\partial^2 \mathfrak{W}}{\partial v_A^2} = -\frac{\partial \mathbf{Y}_A}{\partial v_A}. \tag{17}$$

The latter we can insert in a table of standard operations, analogous with Table XXIII; and we can start a relaxation table, analogous with Table XXIV, by inserting in the first line initial values of the $\mathbf{X}$'s and $\mathbf{Y}$'s as given by (16) when all the u's and v's are zero. The relaxation procedure will be the same as before, and its convergence may be established by an argument exactly similar to that of Chapter V, since by definition (17) the 'quasi-influence coefficients' satisfy reciprocal relations of the type of (8).

121. According to (15)

$$\frac{\partial \mathfrak{W}}{\partial u_A} = \sum_{(j)} \left[\mathbf{X}\frac{\partial \mathbf{X}}{\partial u_A} + \mathbf{Y}\frac{\partial \mathbf{Y}}{\partial u_A}\right], \quad \frac{\partial \mathfrak{W}}{\partial v_A} = \sum_{(j)} \left[\mathbf{X}\frac{\partial \mathbf{X}}{\partial v_A} + \mathbf{Y}\frac{\partial \mathbf{Y}}{\partial v_A}\right], \tag{18}$$

and the differentials in these expressions have constant values typified by (12) of § 118. Hence

$$\left.\begin{aligned} \frac{\partial^2 \mathfrak{W}}{\partial u_A^2} = \sum_{(j)} \left[\left(\frac{\partial \mathbf{X}}{\partial u_A}\right)^2 + \left(\frac{\partial \mathbf{Y}}{\partial u_A}\right)^2\right], \quad \frac{\partial^2 \mathfrak{W}}{\partial v_A^2} = \sum_{(j)} \left[\left(\frac{\partial \mathbf{X}}{\partial v_A}\right)^2 + \left(\frac{\partial \mathbf{Y}}{\partial v_A}\right)^2\right], \\ \frac{\partial^2 \mathfrak{W}}{\partial u_A\, \partial v_B} = \sum_{(j)} \left[\frac{\partial \mathbf{X}}{\partial u_A}\cdot\frac{\partial \mathbf{X}}{\partial v_B} + \frac{\partial \mathbf{Y}}{\partial u_A}\cdot\frac{\partial \mathbf{Y}}{\partial v_B}\right], \quad \ldots, \text{ etc.} \end{aligned}\right\} \tag{19}$$

Combined with (16) and (17) these expressions show that the 'quasi-forces' are linear in the u's and v's and that the 'quasi-influence coefficients' have constant values. We may use them to construct a new table of standard operations (Table XXV) from an operations table of the normal kind (Table XXIII). In the latter, columns and lines are so arranged that the same number relates to a 'corresponding' force and displacement (cf. Chap. I, § 9): thus line 2 relates to a displacement u_C, and column 2 to the residual force $\mathbf{X}_C$. Let the new table be arranged in conformity, so that (for example) line 2 relates to u_C and column 2 to $\mathbf{X}_C$; and suppose that we want to calculate $\partial \mathbf{X}_C/\partial u_B$, the entry appropriate to line 1, column 2 (or, since reciprocal relations are satisfied, to line 2, column 1). Observing that in Table XXIII the entries in lines 1 and 2 of the column appropriate to any force $\mathbf{X}$ are the values of $\partial \mathbf{X}/\partial u_B$ and $\partial \mathbf{X}/\partial u_C$ respectively, we deduce from (19) that for this purpose we

must sum the products of such entries for all columns of Table XXIII which relate to forces included within the summation $\sum_{(j)}$. Such columns are distinguished by heavy vertical rulings in Table XXIII: in Table XXV nothing is gained by retaining other columns, which accordingly have been left blank.

122. Proceeding as described, we have as the appropriate entry in line 1, column 2, and in line 2, column 1, of Table XXV

$$\begin{aligned}\frac{\partial \mathbf{X}_C}{\partial u_B} = \frac{\partial \mathbf{X}_B}{\partial u_C} &= +114{\cdot}37\times 8{\cdot}73+8{\cdot}73\times 20{\cdot}64-1{\cdot}78\times 2{\cdot}55+\\ &\qquad +96{\cdot}44\times 6{\cdot}72+6{\cdot}72\times 1{\cdot}34-6{\cdot}88\times 0{\cdot}55,\\ &= +1{,}827{\cdot}40, \end{aligned} \tag{i}$$

and the other entries are calculated similarly. In our example the work is shortened by the circumstances that not only relations of type (8) are satisfied, but also relations typified by

$$\frac{\partial \mathbf{X}_B}{\partial u_C} = \frac{\partial \mathbf{Y}_B}{\partial v_C}, \quad \frac{\partial \mathbf{X}_B}{\partial v_C} = -\frac{\partial \mathbf{Y}_B}{\partial u_C}, \quad \frac{\partial \mathbf{X}_C}{\partial v_C} = \frac{\partial \mathbf{Y}_C}{\partial u_C} = 0. \tag{20}$$

TABLE XXV

Column number		1	2	3			4	5	6
Number and nature of operation		$\mathbf{X}_B$	$\mathbf{X}_C$	$\mathbf{X}_D$			$\mathbf{Y}_B$	$\mathbf{Y}_C$	$\mathbf{Y}_D$
(1) $u_B = 1$	..	$-22{,}553{\cdot}04$	$1{,}827{\cdot}40$	$925{\cdot}60$	..	..	0	$183{\cdot}79$	$-474{\cdot}91$
(2) $u_C = 1$	..	$1{,}827{\cdot}40$	$-555{\cdot}98$	$55{\cdot}58$	..	..	$-183{\cdot}79$	0	$37{\cdot}61$
(3) $u_D = 1$	..	$925{\cdot}60$	$55{\cdot}58$	$-659{\cdot}90$	..	..	$474{\cdot}91$	$-37{\cdot}61$	0
(4) $v_B = 1$	..	0	$-183{\cdot}79$	$474{\cdot}91$	..	..	$-22{,}553{\cdot}04$	$1{,}827{\cdot}40$	$925{\cdot}60$
(5) $v_C = 1$	..	$183{\cdot}79$	0	$-37{\cdot}61$	..	..	$1{,}827{\cdot}40$	$-555{\cdot}98$	$55{\cdot}58$
(6) $v_D = 1$	..	$-474{\cdot}91$	$37{\cdot}61$	0	..	..	$925{\cdot}60$	$55{\cdot}58$	$-659{\cdot}90$

123. To start the new relaxation table we must insert in its first line the initial values of the **X**'s and **Y**'s. We have from (16) and (18)

$$\mathbf{X}_B \text{ (initial)} = -\sum_{(j)}\left[X\frac{\partial \mathbf{X}}{\partial u_B}+Y\frac{\partial \mathbf{Y}}{\partial u_B}\right], \tag{21}$$

with corresponding expressions for the other 'quasi-forces'; so because the column numberings of Tables XXIII and XXIV conform, if (for example) we want the entry appropriate to column 2 of the new relaxation table, then we are concerned with the first line (initial 'forces') of Table XXIV and with line 2 of Table XXIII. Multiplying together the two numbers which appear in any particular column,

we have to sum the resulting products. In this chosen example the appropriate entry is given by

$$\begin{aligned}\mathbf{X}_C \text{ (initial)} &= -8{\cdot}73\times 101{\cdot}50+20{\cdot}64\times 7{\cdot}58-2{\cdot}55\times 12{\cdot}18- \\ &\qquad -6{\cdot}72\times 79{\cdot}12-1{\cdot}34\times 8{\cdot}99+0{\cdot}55\times 2{\cdot}03 \\ &= -1303{\cdot}32, \qquad\qquad \text{(ii)}\end{aligned}$$

and the other entries are found similarly.

The new relaxation table has not been reproduced, since the procedure required to liquidate these initial **X**'s and **Y**'s is in every way normal. Applied to the example treated already (Fig. 31), fourteen operations sufficed to give the same solution as was found by an unmodified procedure in §§ 117–19.

The 'normalization' of simultaneous equations

124. The device which in §§ 120–3 was employed to circumvent a difficulty associated with alternating currents is applicable to any system of linear simultaneous equations that can be presented, whether arising out of a physical problem or not. We may term it **normalization** of a given set of equations, since the resulting equations have a property of the 'normal equations' which arise in the method of least squares (Ref. 5, Chap. IX)—namely, of being 'axisymmetric'.

Let the given set of n simultaneous equations be written as follows:

$$\left.\begin{aligned} A_1+a_{11}x+a_{12}y+a_{13}z+\ldots &= 0, \\ A_2+a_{21}x+a_{22}y+a_{23}z+\ldots &= 0, \\ A_3+a_{31}x+a_{32}y+a_{33}z+\ldots &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_n+a_{n1}x+a_{n2}y+a_{n3}z+\ldots &= 0. \end{aligned}\right\} \qquad (22)$$

Then, if $a_{21} = a_{12}$, $a_{31} = a_{13}$, $a_{32} = a_{23}$,..., etc. (so that terms which are symmetrically placed with respect to the diagonal drawn in (22) through $a_{11}x$, $a_{22}y$, $a_{33}z$,..., etc. have equal coefficients), we say that the equations are **axisymmetric**. In elastic problems, equalities of this kind are 'Maxwell relations', and the a's (or influence coefficients) are second differentials of a 'total-energy function'. In

general (e.g. in the problem of A.C. networks) Maxwell relations are *not* satisfied, and then we say that the equations are not axisymmetric. But here too 'normalization' gives *derived* equations which are axisymmetric,—their coefficients being differentials of a positive function of the coordinates (namely, the sum of the squares of the 'errors').

In our electrical example it happened that the Maxwell relations of type ($a_{rs} = a_{sr}$) were replaced by 'skew-symmetrical' relations of the type ($a_{rs}+a_{sr} = 0$); but no use was made of these relations in developing the normalization procedure of §§ 120–3, which accordingly would be no less valid if they did not obtain. It follows that we have brought within the scope of relaxation methods *any* system of simultaneous linear equations that can be presented.

The reader will see without difficulty that the modified procedure is equivalent to the following: (i) multiply equations (22) by a_{11}, a_{21},..., a_{n1} respectively, and add to obtain the first derived equation; (ii) multiply them by a_{12}, a_{22},..., a_{n2} respectively, and add to obtain the second derived equation; and so on to obtain n derived equations having the axisymmetric property.

RECAPITULATION

125. This chapter, like the last, is for the most part based on Ref. 1. Its first problem (§ 111) is a straightforward application of processes used already, but its second (§ 116) has suggested a rather important extension of the relaxation technique (§§ 120–3).

Hitherto we have been concerned with systems which in one particular are special,—namely, in that 'Maxwell relations' are satisfied. In the language of mathematics, our problems have been governed by *axisymmetric* equations. It appears in § 115 that this was a factor in our success, which accordingly cannot be guaranteed in relation to equations not of that kind; and our electrical example shows that the restriction has practical as well as theoretical importance. In § 120 it is removed by a process known to mathematicians as the deduction of 'normal' equations (Ref. 5, § 108). We can thus turn to new problems knowing that no set of linear simultaneous equations can be presented which Relaxation Methods are not competent to solve.

The problem of D.C. networks has been treated by Hardy Cross (Ref. 2) as an extension of his Moment Distribution Method.

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VII

THE NATURAL MODES AND FREQUENCIES OF VIBRATING SYSTEMS

I. GENERAL THEORY

The frequency equation

126. HITHERTO we have dealt with problems having the common feature that a unique solution corresponds with every precisely specified set of data. (For example, an elastic framework can assume one and only one configuration of equilibrium under a given system of external forces.) We now consider problems such that a complete solution would define a series of configurations, all satisfying the governing equations but each associated with some particular value of a certain parameter (the 'characteristic number' or *Eigenwert*). Problems of this kind are presented by mechanical or elastic systems executing free vibrations: theory asserts that any free vibration can be regarded as compounded of one or more vibrations in 'normal modes', each associated with a particular frequency, and that the number of mutually independent modes is equal to the number of degrees of freedom by which the system is characterized.

The first section of this chapter provides a short review of underlying principles.

127. When a system has N degrees of freedom, N variable quantities (or **coordinates**) suffice for the specification of any possible configuration. Given their values and the relevant elastic and geometrical constants we can calculate the total potential energy $\mathfrak{V}$ referred to the configuration of statical equilibrium as datum, and given their rates of variation with time we can calculate the total kinetic energy $\mathfrak{T}$. To a first approximation $\mathfrak{V}$ will be a homogeneous quadratic function of the N coordinates and $\mathfrak{T}$ will be a homogeneous quadratic function of their first differentials with respect to time. In any free vibration, conservation of energy requires that

$$\mathfrak{V}+\mathfrak{T} = \text{const. (independent of time).} \tag{1}$$

128. Now in a **normal free vibration** (by definition) every coordinate varies with time in accordance with a common time-factor

$\sin(pt+\epsilon)$, so that (e.g.) the coordinate

$$\left.\begin{aligned} q_k &= a_k \sin(pt+\epsilon), \\ \text{and accordingly} \quad \dot{q}_k &= \frac{dq_k}{dt} = pa_k \cos(pt+\epsilon). \end{aligned}\right\} \tag{2}$$

Therefore we may write

$$\mathfrak{V} = \mathbf{V}\sin^2(pt+\epsilon), \qquad \mathfrak{T} = p^2\mathbf{T}\cos^2(pt+\epsilon), \tag{3}$$

where $\mathbf{V}$, $\mathbf{T}$ are homogeneous quadratic functions of the a's, with coefficients which are calculable when the elastic and geometrical constants of the vibrating system are known. Then the energy equation (1) requires that

$$\mathbf{V} = p^2\mathbf{T}. \tag{4}$$

From this equation the characteristic number p^2 could be determined if the normal mode were known,—that is, the $N-1$ ratios a_2/a_1, a_3/a_1,..., etc. (Since $\mathbf{V}$ and $\mathbf{T}$ are both homogeneous functions of the a's, p^2 does not depend upon the *amplitude* of the vibration.) Actually the mode as well as p^2 must be determined. It is governed by N equations of motion which can be derived in the manner of Lagrange from the expressions for $\mathfrak{V}$ and $\mathfrak{T}$.

129. Lagrange's equations have the form

$$\frac{\partial(\mathfrak{V}-\mathfrak{T})}{\partial q_k}+\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{q}_k}\right) = Q_k \quad (k = 1, 2, 3,..., N)$$

in the general case of motion occurring under the action of external forces typified by Q_k. When (as will be true of *small* free vibrations about a position of stable equilibrium) $\mathfrak{T}$ is a function of the $\dot{q}$'s with coefficients which have constant values independent of the q's, then all quantities of the type $\partial\mathfrak{T}/\partial q_k$ vanish and quantities of the type $\partial\mathfrak{T}/\partial\dot{q}_k$ are linear functions of $\dot{q}_1$, $\dot{q}_2$,..., etc. with constant coefficients, therefore quantities such as $\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{q}_k}\right)$ are linear functions of the $\ddot{q}$'s (i.e. of the accelerations). So for a normal free vibration as defined in (2) the typical Lagrange equation simplifies to

$$\frac{\partial \mathbf{V}}{\partial a_k}-p^2\frac{\partial \mathbf{T}}{\partial a_k} = 0 \quad (k = 1, 2, 3,..., N). \tag{5}$$

Now let $\mathfrak{T}$ and $\mathfrak{V}$ be given by

$$\left.\begin{aligned} \mathfrak{T} &= \tfrac{1}{2}(b_{11}\dot{q}_1^2+b_{22}\dot{q}_2^2+...)+(b_{12}\dot{q}_1\dot{q}_2+...), \\ \mathfrak{V} &= \tfrac{1}{2}(c_{11}q_1^2+c_{22}q_2^2+...)+(c_{12}q_1q_2+...), \end{aligned}\right\} \tag{6}$$

for the system considered. Then, according to (2) and (3),

$$\left.\begin{aligned} \mathbf{T} &= \tfrac{1}{2}(b_{11}a_1^2+b_{22}a_2^2+...)+(b_{12}a_1a_2+...), \\ \mathbf{V} &= \tfrac{1}{2}(c_{11}a_1^2+c_{22}a_2^2+...)+(c_{12}a_1a_2+...), \end{aligned}\right\} \tag{7}$$

so that (5) can be written in the form

$$e_{1k}a_1+e_{2k}a_2+...+e_{kk}a_k+...+e_{Nk}a_N = 0 \quad (k = 1,2,3,...,N), \tag{8}$$

e_{rs} standing for $(c_{rs}-p^2b_{rs})$.

130. Since N equations of type (8) are presented, and since the number of these equations is equal to the number of the unknowns, it would appear at first sight that they are soluble for all values of p^2. But actually, since the equations contain no term independent of the coordinates, there are only $N-1$ unknowns which can vary independently,—namely the ratios a_2/a_1,..., etc. (cf. § 128); and in order that all of equations (8) may be satisfied simultaneously we must have

$$\Delta = \begin{vmatrix} e_{11}, & e_{21}, & . & . & e_{N1} \\ e_{12}, & e_{22}, & . & . & e_{N2} \\ . & . & . & . & . \\ e_{1N}, & e_{2N}, & . & . & e_{NN} \end{vmatrix} = 0. \tag{9}$$

This condition, when we substitute for the e's in terms of the b's and c's from (8), is an equation of the Nth degree in p^2, usually termed the **frequency equation.** It will have (in general) N distinct roots, each representing one natural frequency and associated with a particular mode. Orthodox treatment proceeds from the frequency equation, first calculating its roots in p^2 and then deriving the mode appropriate to each from any $N-1$ of the equations (8). Such treatment is ordinarily difficult if N exceeds 6 or 8.

Rayleigh's principle

131. The N equations of type (5) are the conditions for a stationary value of p^2 as deduced from (4). For if $p^2 = \mathbf{V}/\mathbf{T}$ is stationary, then

$$\delta(\mathbf{V}/\mathbf{T}) = \delta\mathbf{V}/\mathbf{T}-\mathbf{V}\,\delta\mathbf{T}/\mathbf{T}^2 = (\delta\mathbf{V}-p^2\,\delta\mathbf{T})/\mathbf{T}$$

must vanish for all possible variations, therefore for any variation δa_k occurring singly. On this result Lord Rayleigh based his well-known 'principle' whereby the gravest (or lowest) natural frequency may be estimated from (4) on the basis of an assumed form for the corresponding mode:—*A small error made in regard to the mode will, by reason of the stationary property, entail an error of the second order of small quantities in regard to p^2; and in regard to the gravest value*

p_1^2 *(which, being the smallest of the stationary values, must be an absolute minimum) the estimate will err, if at all, on the side of excess.*

The practical value of Rayleigh's principle is now widely recognized. Its physical basis may be stated as follows: Unless the assumed mode is correct, constraints will be needed to maintain it, and the forces on those constraints will depend upon the (imposed) frequency of their fluctuation; but for some particular frequency the forces *on the whole* will do no work, and this is Rayleigh's estimate of the natural frequency.

The conjugate property of normal modes. Normal coordinates

132. The form of a normal mode is determined (§ 128) by $(N-1)$ ratios a_2/a_1, a_3/a_1,..., a_N/a_1, but within this definition the absolute values of a_1, a_2,..., a_N are not restricted. Therefore a normal vibration can be completely specified by attaching values to $N-1$ ratios and to one multiplying quantity which (in conjunction with these values) fixes the absolute value of every coordinate. Since it is a matter of indifference which coordinate is made the denominator in the ratios, we shall represent this (the multiplying quantity) by a special symbol ϕ.

Suppose now that the elastic system is constrained to vibrate in a mode which is a combination of two normal modes. Let the first and second mode be distinguished by suffixes r and s, and let the characteristic numbers associated with them be p_r^2, p_s^2 respectively. Let the multiplying quantities be ϕ_r, ϕ_s, so that in the constrained mode each of the N coordinates has an expression of the form

$$a_k = \phi_r(\alpha_k)_r + \phi_s(\alpha_k)_s$$

in which $(\alpha_k)_r$, $(\alpha_k)_s$ are known ratios of the type of a_2/a_1. Since $\mathbf{V}$ is a homogeneous quadratic function of the coordinates, in relation to the constrained mode its expression will have the form

$$\mathbf{V} = \phi_r^2\mathbf{V}_r + \phi_s^2\mathbf{V}_s + \phi_r\phi_s\mathbf{V}_{rs},$$

$\mathbf{V}_r$, $\mathbf{V}_s$ depending solely on the first and second normal mode, and $\mathbf{V}_{rs}$ consisting of products of the $(\alpha)_r$'s and $(\alpha)_s$'s and so depending on both of the normal modes. Similarly

$$\mathbf{T} = \phi_r^2\mathbf{T}_r + \phi_s^2\mathbf{T}_s + \phi_r\phi_s\mathbf{T}_{rs},$$

$\mathbf{T}_{rs}$ depending on both of the normal modes. Therefore the charac-

teristic number for the constrained mode will be given according to (4) by

$$p^2 = \frac{\phi_r^2 \mathrm{V}_r + \phi_s^2 \mathrm{V}_s + \phi_r \phi_s \mathrm{V}_{rs}}{\phi_r^2 \mathrm{T}_r + \phi_s^2 \mathrm{T}_s + \phi_r \phi_s \mathrm{T}_{rs}} = \frac{N}{D} \quad \text{(say)}, \tag{10}$$

whence we have

$$\begin{aligned} \frac{\partial p^2}{\partial \phi_s} &= \frac{1}{D}\left(\frac{\partial N}{\partial \phi_s} - \frac{N}{D}\frac{\partial D}{\partial \phi_s}\right), \\ &= \frac{1}{D}\left(\frac{\partial N}{\partial \phi_s} - p^2 \frac{\partial D}{\partial \phi_s}\right), \quad \text{by (10) again,} \\ &= \frac{1}{D}[2\phi_s(\mathrm{V}_s - p^2 \mathrm{T}_s) + \phi_r(\mathrm{V}_{rs} - p^2 \mathrm{T}_{rs})]. \end{aligned}$$

In this equation let ϕ_s now tend to zero. Then according to (10) p^2 will tend to the value $\mathrm{V}_r/\mathrm{T}_r = p_r^2$, and D to the value $\phi_r^2 \mathrm{T}_r$. So, when $\phi_s = 0$, we have

$$\frac{\partial p^2}{\partial \phi_s} = \frac{\mathrm{V}_{rs} - p_r^2 \mathrm{T}_{rs}}{\phi_r \mathrm{T}_r}.$$

But now, since $\phi_s = 0$, the mode is normal and accordingly (§ 131) p^2 is stationary for all possible variations of the mode. Therefore $\partial p^2/\partial \phi_s$ must be zero—which means (since ϕ_r and T_r are finite) that

$$\mathrm{V}_{rs} = p_r^2 \mathrm{T}_{rs}.$$

By a similar argument we could have shown that

$$\mathrm{V}_{rs} = p_s^2 \mathrm{T}_{rs},$$

and from the last two equations it follows that

$$\mathrm{V}_{rs} = \mathrm{T}_{rs} = 0, \quad \text{if } p_r^2 \neq p_s^2. \tag{11}$$

133. This result expresses what is termed the **conjugate property** of normal modes: *their contributions to* **V** *are additive, also their contributions to* **T**. We have said (§ 126) that any possible mode of vibration can be regarded as made up of one or more normal vibrations: as an equivalent statement we may say that the N multipliers $\phi_1, \phi_2, \ldots$, etc. constitute a possible system of coordinates, in that (when the normal modes are known) any possible mode of vibration can be specified by assigning values to them. They are commonly termed **normal coordinates**.

In virtue of the conjugate property, when normal coordinates are employed the general expression for p^2 takes a form similar to (10) with V_{rs} and T_{rs} suppressed, viz.

$$p^2 = \frac{\phi_1^2 \mathrm{V}_1 + \phi_2^2 \mathrm{V}_2 + \phi_3^2 \mathrm{V}_3 + \ldots}{\phi_1^2 \mathrm{T}_1 + \phi_2^2 \mathrm{T}_2 + \phi_3^2 \mathrm{T}_3 + \ldots} = \frac{N}{D} \quad \text{(say)}. \tag{12}$$

When all the ϕ's vanish except one (ϕ_r, say) this reduces to

$$p^2 = p_r^2 = \frac{\mathbf{V}_r}{\mathbf{T}_r}, \tag{13}$$

in agreement with (4).

Extension of Rayleigh's principle

134. The practical value of the conjugate relations lies in the general formula (12) which has been based upon them. According to that formula

$$\frac{\partial p^2}{\partial \phi_r} = 2\phi_r\left(\frac{\mathbf{V}_r}{D} - N\frac{\mathbf{T}_r}{D^2}\right) = 2\frac{\phi_r}{D}(\mathbf{V}_r - p^2\mathbf{T}_r),$$

$$= 0, \quad \text{by (13), when all the } \phi\text{'s vanish except } \phi_r.$$

This shows (in accordance with § 131) that p^2 has a stationary value when the mode is normal, and it follows that p_1^2 is the minimum value which can be assumed by p^2 ('Rayleigh's principle'). But the expression (12), by which (4) is replaced when normal coordinates are employed, shows that if ϕ_1 is zero then the minimum value of p^2 is p_2^2. That is to say, *p_2^2 is the minimum value of p^2 as given by* (4) *provided that the mode has no 'first normal component'*.

135. Suppose that an assumed mode satisfies this requirement. Then the corresponding quantities **V** and **T**, expressed in terms of normal coordinates, will have the forms

$$\mathbf{V} = \phi_2^2\mathbf{V}_2 + \phi_3^2\mathbf{V}_3 + \ldots,$$
$$\mathbf{T} = \phi_2^2\mathbf{T}_2 + \phi_3^2\mathbf{T}_3 + \ldots,$$

and so, for the assumed mode,

$$\frac{\partial \mathbf{V}}{\partial \phi_1} = \frac{\partial \mathbf{T}}{\partial \phi_1} = 0$$

whatever be the magnitudes of ϕ_2, ϕ_3,..., etc. This means that no work, *on the whole*, will be done either by the elastic or kinetic forces of the assumed mode in acting through displacements corresponding with the first normal mode (ϕ_1); and the Reciprocal Theorem (*Elasticity* § 12) shows that in consequence *no work would be done by the elastic or kinetic forces of the first mode in acting through the displacements of the assumed mode.* We thus obtain a relation which must hold between displacements in an assumed mode in order that this shall have no 'first normal component'; and in regard to modes thus

restricted Rayleigh's principle (§ 131) can be applied with the modification that p_2^2 is now the minimum value which can be assumed by p^2 as calculated from (4). Clearly the argument can be extended.

136. To formulate the relation, we now denote by $(V_k)_1$, $(T_k)_1$ the typical elastic and kinetic force, respectively, of the first mode, V_k, T_k replacing (for brevity) the partial differentials $\partial\mathbf{V}/\partial a_k$, $\partial\mathbf{T}/\partial a_k$. Then, if a_k stands for the typical displacement in the assumed mode, the above criterion requires that

$$\left.\begin{aligned} a_1(V_1)_1+a_2(V_2)_1+...+a_N(V_N)_1 &= 0, \\ a_1(T_1)_1+a_2(T_2)_1+...+a_N(T_N)_1 &= 0, \end{aligned}\right\} \tag{14}$$

these relations being equivalent because the ratio $(T_k)_1/(V_k)_1$ has the same value, by (5), for all values of k. Before (14) can be utilized the first mode must, of course, be known with certainty: this is where Relaxation Methods have an advantage in that they can calculate closely not only p_1^2 but also the associated mode (which Rayleigh's principle does not determine).

Assuming, then, that the V and T forces of the first mode can be found to a close approximation, we have in (14) a linear relation between the a's with known numerical coefficients; and using this to express any one of the displacements (a_1, say) in terms of the others, by substitution in (7) we shall obtain new expressions for $\mathbf{V}$ and $\mathbf{T}$ in which a_1 does not appear. They will hold generally in respect of any mode which contains no 'first normal component', and as such may be made the starting-point of a new calculation, based on Rayleigh's principle, to determine p_2^2. Knowing p_1^2 and p_2^2 we can similarly calculate p_3^2, and so on.

137. A physical interpretation can be given to (14). In § 131 we contemplated a constraint by which a specified mode and frequency could be imposed without restriction of amplitude,—that is, a constraint controlling the relative magnitudes of the different displacements: similarly we can contemplate a constraint imposed to control relative magnitudes in accordance with (14). Either constraint may be visualized as a system of levers by which a_1 is compelled to assume a value determined by the other displacements; but whereas in § 131 the constraint on the whole did work unless the imposed frequency had a certain value (Rayleigh's approximation to the normal frequency), *in this instance no work will be done at any frequency.*

This is evident when we consider the nature of the lever system in each of the two cases. If a specified mode is to be imposed, the system of levers will be as shown in Fig. 32 *a*, and it will cause every coordinate to be determined by the displacement of A: consequently the order of the freedom is reduced to 1, and it is only at A that external force need be applied; but this force must supply at A the

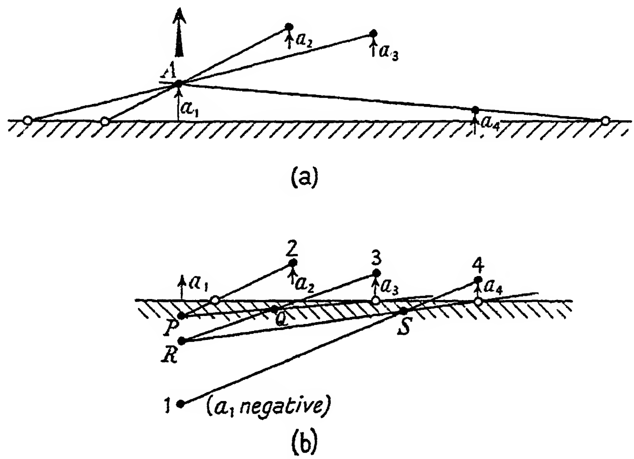

FIG. 32

work which is done at other points by the various levers, and the total work supplied will not in general vanish. When on the other hand (14) is to be satisfied, the lever system will be of the kind shown diagrammatically in Fig. 32 *b*, and it will (as drawn) determine a_1 in conformity with that equation for any imposed values of a_2, $a_3, \ldots, a_N$. Because the joints P, Q,... are in equilibrium, it is easy to show (by the theory of levers) that the forces imposed on 1, 2,..., N will in all circumstances be proportional to $(T_1)_1$, $(T_2)_1, \ldots, (T_N)_1$; and as such they will *on the whole* do no work (i.e. their effect will not enter into the energy equation) because the mode can have no first normal component.

138. We shall not, of course, know the first mode exactly; therefore in satisfying (14) we shall not *completely* eliminate the first component. Consequently the frequency as calculated by Rayleigh's method will be expressed by (12) with ϕ_1 not suppressed but made very small, and strictly speaking the minimum permitted value of p^2 will still be p_1^2. But for practical purposes this will not matter,

because the modifications required to make the mode conform with the first mode would bring all displacements very nearly to zero, and in computation it is easy to guard against this possibility.

Rayleigh's theorem regarding the effect of added mass or constraint

139. In this chapter we shall utilize a further application of the stationary property, also due to Lord Rayleigh (cf. Ref. 4, § 6), whereby information is given regarding the effect on the natural frequencies of a change made either in the masses or in the restoring forces of an elastic system. We shall refer to this as 'Rayleigh's second theorem': *Excepting particular cases in which the effect is nil* (as when mass is added at a nodal point), *any increase of mass will lower the natural frequencies, any increase of stiffness will raise them, and vice versa.*

To prove the theorem we conceive the change to be made in a series of infinitesimal steps, and it is here that use is made of the stationary property. Any change made in the mass or stiffness of a system will (in general) alter both the mode and the associated frequency of every free vibration: therefore the whole effect on any frequency of an infinitesimal change may be regarded as consisting (i) of the effect as calculated without allowance for the accompanying alteration in the mode and (ii) of the effect of this latter alteration. But, in virtue of the stationary property (§ 131), an infinitesimal change in the mode will entail a change of the second order in p^2, consequently in any one of the infinitesimal steps, and therefore in the integrated result of all such steps, (ii) is negligible in comparison with (i). Now the sign of (i) can be deduced from (4), since if the mode is unaltered $\mathbf{T}$ must be increased (and $\mathbf{V}$ left unchanged) by any increase of mass, while $\mathbf{V}$ must be increased (and $\mathbf{T}$ left unchanged) by any increase of stiffness in the system. Hence we have the theorem stated.

'Relaxation of constraints' in relation to vibrating systems

140. Let us suppose that constraints are operative which permit the maintenance of any specified displacements whether steady or oscillatory, and that we use them to maintain a trial solution in which the N displacements of type q_k all vary in accordance with (2). Now writing

$$\left.\begin{array}{l} \mathbf{F}_k = p^2 T_k - V_k, \\ \text{where } T_k \text{ and } V_k \text{ (for brevity) replace } \dfrac{\partial \mathbf{T}}{\partial a_k} \text{ and } \dfrac{\partial \mathbf{V}}{\partial a_k}, \end{array}\right\} \quad (15)$$

we can interpret $\mathbf{F}_k \sin(pt+\epsilon)$ as the force, corresponding with the generalized displacement q_k, which comes upon the constraint controlling q_k. Then, disregarding the common time-factor $\sin(pt+\epsilon)$, we may think of $\mathbf{F}_k$ as the force 'corresponding with' a displacement a_k.

Multiplying (15) throughout by $\frac{1}{2}a_k$, and summing the N equations which can be obtained in this way, we have

$$\tfrac{1}{2}\sum_N [a_k \mathbf{F}_k] = p^2\mathbf{T}-\mathbf{V}, \tag{16}$$

in virtue of the relations

$$\left.\begin{aligned} 2\mathbf{V} &= a_1\frac{\partial \mathbf{V}}{\partial a_1}+a_2\frac{\partial \mathbf{V}}{\partial a_2}+\ldots+a_k\frac{\partial \mathbf{V}}{\partial a_k}+\ldots, \\ 2\mathbf{T} &= a_1\frac{\partial \mathbf{T}}{\partial a_1}+a_2\frac{\partial \mathbf{T}}{\partial a_2}+\ldots+a_k\frac{\partial \mathbf{T}}{\partial a_k}+\ldots, \end{aligned}\right\} \tag{17}$$

which state a familiar property of quadratic forms. From (16) equation (4) results provided that $\sum_N [a_k \mathbf{F}_k] = 0$; and observing that $\frac{1}{2}\sum_N [a_k \mathbf{F}_k]$ measures the total work done by the forces on the constraints in virtue of the a-displacements, we find in this result confirmation of the physical interpretation given in § 131.

141. When $\mathbf{F}_k$ is thus regarded as the 'residual force' on a constraint, the 'influence coefficients' will be given by such quantities as

$$\left.\begin{aligned} \widehat{kk} &= \frac{\partial \mathbf{F}_k}{\partial a_k} = p^2\frac{\partial T_k}{\partial a_k}-\frac{\partial V_k}{\partial a_k}, \\ \widehat{kr} &= \frac{\partial \mathbf{F}_k}{\partial a_r} = p^2\frac{\partial T_k}{\partial a_r}-\frac{\partial V_k}{\partial a_r} = \frac{\partial \mathbf{F}_r}{\partial a_k} = \widehat{rk}, \end{aligned}\right\} \tag{18}$$

in virtue of (15). Their expressions involve p^2, and on that account their values will require to be altered as liquidation proceeds: for the rest they involve quantities of the types

$$\left.\begin{aligned} &\frac{\partial T_k}{\partial a_k} = \frac{\partial^2 \mathbf{T}}{\partial a_k^2}, \qquad \frac{\partial V_k}{\partial a_k} = \frac{\partial^2 \mathbf{V}}{\partial a_k^2}, \\ &\frac{\partial T_k}{\partial a_r} = \frac{\partial^2 \mathbf{T}}{\partial a_r\,\partial a_k}, \qquad \frac{\partial V_k}{\partial a_r} = \frac{\partial^2 \mathbf{V}}{\partial a_r\,\partial a_k}, \end{aligned}\right\} \tag{19}$$

which, since they have fixed values, can be calculated once for all when $\mathbf{V}$ and $\mathbf{T}$ are given as functions of the a's.

Then, starting from a guess (as good as possible) as to the form of the first mode, we can deduce the corresponding forces of types

V_k and T_k and hence, using (17) and (4), a corresponding estimate of p_1^2. Inserting this value in (15) we can evaluate the forces which come initially on the constraints; inserting it in (18), from the table already constructed we can deduce starting-values for the influence coefficients; and embodying these starting-values in an 'operations table' we can start by a normal relaxation process to liquidate the initial forces. When every force has been brought nearly to zero, the altered values of $a_1, a_2, \ldots, a_N$ will represent a closer approximation to the normal mode. *We shall not have solved our problem,* because although the mode as thus modified would entail no appreciable forces on constraints if p^2 had the value adopted in our calculations, it is not consistent (in general) with that value according to (4).

142. But now we can take advantage of the minimal property of p_1^2, which means (§ 131) that starting from a reasonably close assumption for the first normal mode, according to (4) we shall start with a very close approximation to p_1^2 and thereafter (since we shall be working in the neighbourhood of its minimum) p^2 as given by (4) will alter very slowly as the a's are modified. Hence, according to (15), the **F**'s will be very little affected whether p^2 is kept constant throughout the liquidation process or is altered in accordance with (4) after each operation. Therefore we may conveniently work in 'stages' during each of which p^2 is kept constant, and alter its value only at the end of every stage, when all residual forces have been brought (sensibly) to zero.

According to (18) and (19)

$$\widehat{kk} = \frac{2}{a_k^2}(p^2\mathbf{T}-\mathbf{V})$$

when **T** *and* **V** *have values corresponding with a single displacement* a_k. By Rayleigh's principle, in these circumstances $\mathbf{V}/\mathbf{T}$ will have a value not less (and as a rule considerably greater) than p_1^2: consequently throughout the liquidation process, unless p^2 very seriously overestimates p_1^2, all influence coefficients of the type $\widehat{kk}$ will have negative values, therefore any residual force can be brought (temporarily) to zero by the imposition of a 'corresponding' displacement. Moreover it has been shown in (18) that

$$\widehat{kr} = \widehat{rk},$$

so the influence coefficients satisfy 'Maxwell relations' exactly as in a statical problem: consequently the usual proof of convergence (§§ 93–7) will hold in respect of any one stage of the relaxation process, during which (p^2 being treated as constant) all entries in the operations table have constant values.

No modification of the normal relaxation procedure is required except this division into stages and a recalculation at the end of every stage of p^2, of the residual forces, and of the operations table.

Procedure for the imposition of upper and lower limits on the gravest frequency

143. Our methods will not give p_1^2 exactly, but using theorems which have been stated earlier we can at any stage 'bracket' its value between upper and lower limits which will approach more and more closely as the liquidation process is continued.

The upper limit can be found by Rayleigh's principle, which requires (§ 140) that

$$\sum [a_k \mathbf{F}_k] = 0. \tag{20}$$

According to (15) an addition Δp^2 made to a trial value of p^2 will alter $\mathbf{F}_k$ to $(\mathbf{F}_k + \Delta p^2 T_k)$, and the altered values will satisfy (20) if

$$\Delta p^2 = -\sum [a_k \mathbf{F}_k] / \sum [a_k T_k]. \tag{21}$$

144. A lower limit can be based on 'Rayleigh's second theorem' (§ 139). Any trial solution (i.e. mode) can be made exact if appropriate changes are made to the masses, and according to the theorem a chosen value of p^2 will *underestimate* the frequency of a given system if it applies exactly to a system modified by *addition* of mass: therefore the highest value of p^2 which can with certainty be termed a lower limit is that value which requires, to make it hold exactly in respect of the mode considered, no change in one mass and positive additions to all the others.

In an exact solution (p_1^2 having its correct value) every $\mathbf{F}$ is zero in accordance with (5): in a trial solution, any one force can be made to vanish by a suitable choice of frequency, but a different value (in general) is required for each of the several forces. On the other hand, whatever value be attached to p_1^2 we can make $\mathbf{F}_k$ zero by suitably altering the value of T_k ($= \partial \mathbf{T}/\partial a_k$), and this alteration will not affect the other forces if it is made by merely changing the coefficient of a_k^2 in the quadratic expression for $\mathbf{T}$. Increasing that

coefficient we shall be adding mass to the system, and vice versa. (The added mass will increase the kinetic energy when, and only when, $\dot{q}_k$ is non-zero.)

If when every $\mathbf{F}$ has been brought to zero every mass has been either left unaltered or increased, then the chosen value of p^2 will be exact as regards a system modified by *addition* of mass and will therefore (by the theorem of § 139) *underestimate* the frequency of the unmodified (i.e. the given) system. In other words, it will furnish a lower limit to the required value p_1^2. Now an increase made in the mass associated with $\dot{q}_k$ will entail a positive increase in the value of $a_k T_k$ and therefore, by (15), a positive increase in the value of $a_k \mathbf{F}_k$. Consequently only positive or zero additions of mass will be needed to bring all $\mathbf{F}$'s to zero *provided that every product of the type* $a_k \mathbf{F}_k$ *is negative or zero*, and on that understanding we may assert that the chosen value of p^2 is less than p_1^2.

145. The highest value of p^2 which can with certainty be termed a lower limit is the highest value which makes one of the $a.\mathbf{F}$ products zero and all the others negative.

Suppose now that at some stage in the liquidation process all products of the type $a_k \mathbf{F}_k$ have been calculated for some trial value p^2, and let p^2 be altered to $(p^2+\Delta p^2)$. By (15) the value of $a_k \mathbf{F}_k$ will be altered to

$$a_k(\mathbf{F}_k+\Delta p^2 T_k), \tag{22}$$

and this altered value will be zero if

$$\Delta p^2 = -\frac{\mathbf{F}_k}{T_k}. \tag{23}$$

An algebraically greater value of Δp^2 will make the altered value positive provided that $a_k T_k$ is positive. If, then, $(\mathbf{F}/T)_L$ denotes *the algebraically largest value of* $\mathbf{F}_k/T_k$ *which corresponds with a positive product* $a_k T_k$, we may assert that

$$p_1^2 \nless p^2-\left(\frac{\mathbf{F}}{T}\right)_L, \tag{24}$$

but no higher value can be imposed (by our argument) as a lower limit.

Provided that the relaxation procedure has been continued until all of the $\mathbf{F}$'s are small, the lower limit given by (24) will always be positive: it may happen that $(\mathbf{F}/T)_L$ is negative because too low a

value has been assumed for p_1^2, but the formula will not be invalidated on that account. Negative values of $a_k T_k$ must be contemplated, and strictly speaking when $a_k T_k$ is negative $a_k \mathbf{F}_k$ will change sign *from positive to negative* as Δp^2 increases through the value given by (23), therefore (24) will be invalidated unless $\mathbf{F}_k/T_k$ is algebraically *greater* than $(\mathbf{F}/T)_L$ as defined above. But negative values of $a_k T_k$ will occur (if at all) only in the neighbourhood of nodal points, and there (for practical purposes) they may be neglected.

146. The converse theorem to that of § 144 (that p^2 in (15) may be regarded as an upper limit when every product of the type $a_k \mathbf{F}_k$ is zero or *positive*) can be established and applied in similar fashion; but its practical value is less, in that the upper limit imposed by Rayleigh's principle is lower and therefore preferable. (We have proved this in § 140, where it was shown that the 'Rayleigh upper limit' makes $\sum_N [a_k \mathbf{F}_k]$ zero; for when that condition is realized one or more products of the type $a_k \mathbf{F}_k$ will evidently be negative, unless the trial solution should happen to be exact.)

Accordingly we now combine (21) and (24) in the inequality

$$p^2 - \left(\frac{\mathbf{F}}{T}\right)_L < p_1^2 < p^2 - \frac{\sum_N [a_k \mathbf{F}_k]}{\sum_N [a_k T_k]}, \tag{25}$$

which can be used at any stage of the liquidation process to impose limits on the value of p_1^2. As stated previously (§ 143), if the process is performed correctly the limits will continually approach, and it is thus easy to decide when computation should stop.

A numerical example: Torsional vibrations of an elastic shaft

147. To illustrate this account of principles and methods we shall consider torsional vibrations of the system shown in Fig. 33. A light shaft of uniform torsional rigidity C, carrying six disks, pulleys or similar inertial loads, has one end O held fixed and so can vibrate in modes which have a node (i.e. a stationary point) at that end. Our problem is to determine the natural frequencies and the mode associated with each.

It should be emphasized that here and in every problem relating to vibrations there is in fact no necessity to define the physical

system thus exactly,—all that matters is the formulation of **V** and **T**. We have taken a torsional system in order to fix ideas; but equations (28) can equally be regarded as relating to transverse vibrations of

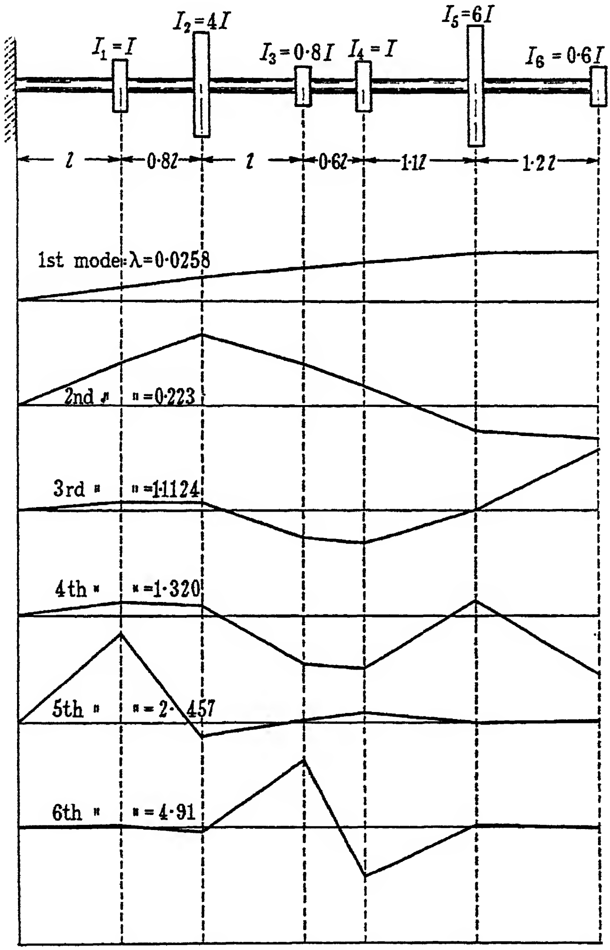

FIG. 33

a loaded string, or (with appropriate changes in the meaning of the coordinates) to any kind of vibrating system. Cf. § 177 (Chap. IX).

148. To obviate the inconvenience of dimensional factors, and at the same time give our solutions somewhat greater generality, at the outset we express all inertial loads as multiples of I_1 and all shaft lengths as multiples of l (cf. Fig. 33). Then, if $\alpha\times(1, a_2,..., a_6)$ are

the angular displacements of $I_1, I_2,\ldots, I_6$ respectively, α being proportional to $\sin(pt+\epsilon)$, it is easy to verify that

$$\mathbf{V} = \frac{1}{2}\frac{C}{l}\alpha^2\left[1+\frac{(a_2-1)^2}{0\cdot 8}+\frac{(a_3-a_2)^2}{1}+\frac{(a_4-a_3)^2}{0\cdot 6}+\right.$$
$$\left.+\frac{(a_5-a_4)^2}{1\cdot 1}+\frac{(a_6-a_5)^2}{1\cdot 2}\right],$$
$$\mathbf{T} = \tfrac{1}{2}I_1\,\alpha^2[1+4a_2^2+0\cdot 8a_3^2+a_4^2+6a_5^2+0\cdot 8a_6^2].$$

Now introducing a non-dimensional quantity

$$\lambda = p^2 I_1\, l/C, \tag{26}$$

and substituting in (4), we see that the dimensional factors in $\mathbf{V}$ and $\mathbf{T}$ (namely $C\alpha^2/l$ and $I_1\alpha^2$) may be suppressed provided that p^2 is replaced by λ. Making this simplification, hereafter we shall take as expressions for $\mathbf{V}$ and $\mathbf{T}$

$$\left.\begin{aligned} 2\mathbf{V} &= 2\cdot 25(1+a_2^2)+2\cdot\dot{6}a_3^2+2\cdot\dot{5}\dot{7}a_4^2+1\cdot 7\dot{4}\dot{2}a_5^2+0\cdot 8\dot{3}a_6^2-\\ &\qquad -2(1\cdot 25a_2+a_2a_3+1\cdot\dot{6}a_3a_4+0\cdot\dot{9}\dot{0}a_4a_5+0\cdot 8\dot{3}a_5a_6),\\ 2\mathbf{T} &= 1+4a_2^2+0\cdot 8a_3^2+a_4^2+6a_5^2+0\cdot 8a_6^2, \end{aligned}\right\} \tag{27}$$

and we shall write instead of (5), (4) and (15)

$$V_k-\lambda T_k = 0, \tag{5a}$$
$$\mathbf{V} = \lambda\mathbf{T}, \tag{4a}$$
$$\mathbf{F}_k = \lambda T_k-V_k, \tag{15a}$$

V_k, T_k standing as before for $\partial\mathbf{V}/\partial a_k$, $\partial\mathbf{T}/\partial a_k$. In (27) the a's may be regarded as purely numerical.

149. Equation (5a) must hold for $k = 2, 3, 4, 5, 6$, a_1 $(=1)$ being now invariant. So, according to (27), we have

$$\left.\begin{aligned} 2\cdot 25a_2-(1\cdot 25+a_3) &= \lambda\times 4a_2,\\ 2\cdot\dot{6}a_3-(a_2+1\cdot\dot{6}a_4) &= \lambda\times 0\cdot 8a_3,\\ 2\cdot\dot{5}\dot{7}a_4-(1\cdot\dot{6}a_3+0\cdot\dot{9}\dot{0}a_5) &= \lambda a_4,\\ 1\cdot 7\dot{4}\dot{2}a_5-(0\cdot\dot{9}\dot{0}a_4+0\cdot 8\dot{3}a_6) &= \lambda\times 6a_5,\\ 0\cdot 8\dot{3}(a_6-a_5) &= \lambda\times 0\cdot 8a_6. \end{aligned}\right\} \tag{28}$$

Multiplying these equations by $a_2, a_3,\ldots, a_6$ respectively and summing, we deduce that if $\mathbf{V} = \lambda\mathbf{T}$, then

$$2\cdot 25-1\cdot 25a_2 = \lambda, \tag{29}$$

and this equation is what we should have obtained from (5a) with $k = 1$, if a_1 had been left as a variable in (27) and made unity after

differentiation. Thus no generality has been lost by our 'non-dimensional' treatment.

Equations (28) and (29) permit an easy verification of results obtained in Chapter VIII by relaxation methods. Giving any value to λ we can deduce a_2 from (29), then a_3 from the first of (28), a_4 from the second, a_5 from the third, and a_6 from the fourth. The fifth of (28), unless by chance, will not be satisfied; but plotting the ratio of its left- and right-hand sides against λ we can use the resulting curve to estimate values of λ for which this ratio is unity.† Here the ratio is found to change from a value slightly on one side of unity to a value slightly on the other side within the following ranges of λ:

$$\left.\begin{array}{ll}
 & \textit{Fractional range} \\
 & \textit{of uncertainty} \\
\lambda_1 = 0{\cdot}02585604 \pm 0{\cdot}00000475 & \pm 0{\cdot}018_4\% \\
\lambda_2 = 0{\cdot}2239265 \pm 0{\cdot}0000430 & \pm 0{\cdot}019_{25}\% \\
\lambda_3 = 1{\cdot}1124143 \pm 0{\cdot}0000103 & \pm 0{\cdot}00092_6\% \\
\lambda_4 = 1{\cdot}3205877 \pm 0{\cdot}0000547 & \pm 0{\cdot}0041_4\% \\
\lambda_5 = 2{\cdot}457850 \pm 0{\cdot}000142 & \pm 0{\cdot}0057_8\% \\
\lambda_6 = 4{\cdot}914422 \pm 0{\cdot}001613 & \pm 0{\cdot}032_8\%
\end{array}\right\} \quad (30)$$

Moreover, the values thus obtained for $a_1, a_2, \ldots, a_6$ confirm very satisfactorily the modes as determined in Chapter VIII. These modes are indicated in Fig. 33: their forms are not such as could have been estimated *a priori*.

150. The orthodox procedure, in which the first step is the formulation of a frequency equation, was described in § 130. In this instance all six of (28) and (29) must be satisfied, so that eliminating $a_2, a_3, \ldots, a_6$ we have

$$\begin{vmatrix}
2{\cdot}25-\lambda, & -1{\cdot}25, & 0, & 0, & 0, & 0 \\
-1{\cdot}25, & 2{\cdot}25-4\lambda, & -1, & 0, & 0, & 0 \\
0, & -1, & 2{\cdot}\dot{6}-0{\cdot}8\lambda, & -1{\cdot}\dot{6}, & 0, & 0 \\
0, & 0, & -1{\cdot}\dot{6}, & 2{\cdot}\dot{5}\dot{7}-\lambda, & -0{\cdot}\dot{9}\dot{0}, & 0 \\
0, & 0, & 0, & -0{\cdot}\dot{9}\dot{0}, & 1{\cdot}7\dot{4}\dot{2}-6\lambda, & -0{\cdot}8\dot{3} \\
0, & 0, & 0, & 0, & -0{\cdot}8\dot{3}, & 0{\cdot}8\dot{3}-0{\cdot}8\lambda
\end{vmatrix} = 0, \quad (31)$$

—an equation of the sixth degree in λ. This yields six natural frequencies, each associated with a particular mode.

† This is described by Den Hartog (Ref. 1, § 40) as 'Holzer's method'.

No occasion for this elimination of displacements arises in the orthodox treatment of any problem considered hitherto, for the reason that the governing equations (corresponding with (28) of § 149) include constant terms depending on specified external 'forces', consequently are in fact equal in number to the unknown displacements. Evidently the class of problem now under consideration presents difficulties of quite different order.

We shall not pursue the orthodox method further. Instead, we proceed (in Chapter VIII) to apply to this example the alternative methods which have been outlined in §§ 140–6.

RECAPITULATION

151. In this chapter an attack is started on problems of a kind quite different from those which have engaged attention in Chapters I–VI. A new technique is necessary, and this in turn entails a new examination of convergence.

A brief account of general vibration theory (§§ 126–39) leads by way of two theorems due to Rayleigh (Ref. 3) to foundations for a relaxation treatment of 'characteristic number' (*Eigenwert*) problems. Most of the matter comes from a recent paper (Ref. 4), but a numerical example is presented and solved by orthodox methods (§§ 147–50) in order that the new method may be put to quantitative test in Ch..pter VIII.

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VIII

THE NATURAL MODES AND FREQUENCIES OF VIBRATING SYSTEMS

II. PRACTICAL DETAILS OF THE RELAXATION TECHNIQUE

152. Attacking the problem of § 147 by relaxation methods, we begin by constructing Table XXVI, deriving the relevant quantities from (27) in accordance with § 141.

By including columns and a line for $k = 1$, $r = 1$, we can take advantage in computation of the relations (17). For all quantities of the types $V_k = \partial\mathbf{V}/\partial a_k$, $T_k = \partial\mathbf{T}/\partial a_k$ are linear functions of the a's without constant terms, so can be computed with the aid of Table XXVI when values have been assigned to $a_1, a_2,..., a_6$: then, multiplying by the a's and summing in accordance with (17), we can evaluate $\mathbf{V}$, $\mathbf{T}$ and deduce a corresponding value of λ from (4 a) of § 148. This method of computing $\mathbf{V}$, $\mathbf{T}$, λ is given tabular form in Table XXVII. We keep a_1 unity throughout (cf. §§ 148–9).

In order that the test of method may be stringent, we make no attempt in this instance to guess the first mode closely. Instead we assume a roughly uniform twist of the shaft, taking as our starting assumption

$$(a_1 = 1), \quad a_2 = 2, \quad a_3 = 3, \quad a_4 = 4, \quad a_5 = 5, \quad a_6 = 6. \quad (32)\dagger$$

Table XXVII relates to these values and deduces corresponding values of $\mathbf{T}$, $\mathbf{V}$ and λ $(= \mathbf{V}/\mathbf{T})$. We know that the value (0·0304068) of λ must be too high, so we proceed on the starting assumption that

$$\lambda = 0{\cdot}03, \tag{33}$$

and from Table XXVI, giving λ this value, we construct an operations table (Table XXVIII) for use in the first stage of a relaxation process.

To facilitate checking by the reader Tables XXVI–XXVIII have been made *exact* by the retention of recurring decimals. Further (since the example is to be regarded less as a physical problem than as a test of method) more significant figures have been retained throughout this chapter than would ordinarily be reasonable, having regard to the uncertainty in practice of the physical data. Correctly

† To facilitate reference the equations of this chapter have been numbered consecutively with those of Chapter VII.

TABLE XXVI. *Values of $\partial^2\mathbf{T}/\partial a_r\,\partial a_k$ and of $\partial^2\mathbf{V}/\partial a_r\,\partial a_k$ for the System defined in Fig.* 33

	$\partial^2\mathbf{T}/\partial a_r\,\partial a_k$						$\partial^2\mathbf{V}/\partial a_r\,\partial a_k$					
$k =$	1	2	3	4	5	6	1	2	3	4	5	6
$r =$ 1	1	..	..	..	..	..	2·25	−1·25	..	..	..	..
2	..	4	..	..	..	..	−1·25	2·25	−1	..	..	..
3	..	..	0·8	..	..	..	..	−1	2·6	−1·6	..	..
4	..	..	..	1	..	..	..	..	−1·6	2·57	−0·90	..
5	..	..	..	..	6	..	..	..	..	−0·90	1·742	−0·83
6	..	..	..	..	..	0·8	..	..	..	..	−0·83	0·83

TABLE XXVII

	$T_k = \partial\mathbf{T}/\partial a_k$						$V_k = \partial\mathbf{V}/\partial a_k$					
$k =$	1	2	3	4	5	6	1	2	3	4	5	6
$a_1 = 1$	1	0	0	0	0	0	2·25	−1·25	0	0	0	0
$a_2 = 2$	0	8	0	0	0	0	−2·5	4·5	−2	0	0	0
$a_3 = 3$	0	0	2·4	0	0	0	0	−3	8	−5	0	0
$a_4 = 4$	0	0	0	4	0	0	0	0	−6·6	10·30	−3·63	0
$a_5 = 5$	0	0	0	0	30	0	0	0	0	−4·54	8·712	−4·16
$a_6 = 6$	0	0	0	0	0	4·8	0	0	0	0	−5	5
Total forces	1	8	2·4	4	30	4·8	−0·25	0·25	−0·6	0·76	0·075	0·83
a_k	1	2	3	4	5	6	1	2	3	4	5	6
Products	1	16	7·2	16	150	28·8	−0·25	0·5	−2	3·03	0·378	5

Total $= 2\mathbf{T} =$ *219* $(\lambda = \mathbf{V}/\mathbf{T} =$ *0·0304068*$)$ Total $= 2\mathbf{V} =$ 0·6590

TABLE XXVIII. *Operations Table for the System defined in Fig.* 33: *First Stage of Relaxation Process.* $(\lambda = 0{\cdot}03)$

No. and nature of operation		$\mathbf{F}_1$	$\mathbf{F}_2$	$\mathbf{F}_3$	$\mathbf{F}_4$	$\mathbf{F}_5$	$\mathbf{F}_6$	$\Sigma\,(\mathbf{F})$
2	$a_2 = 1{,}000$	1,250	−2,130	1,000	..	..	..	120
3	$a_3 = 1{,}000$	..	1,000	−2,642·6	1,666·6	..	..	24
4	$a_4 = 1{,}000$	..	..	1,666·6	−2,545·75	909·00	..	30
5	$a_5 = 1{,}000$	..	..	..	909·00	−1,562·42	833·3	180
6	$a_6 = 1{,}000$	..	..	..	..	833·3	−809·3	24
1	$a_2 = a_3 = a_4 = a_5 = a_6 = 1{,}000$	1,250	−1,130	24	30	180	24	378

to eight significant figures, according to (15a) with $\lambda = 0{\cdot}03$ we have from the line marked 'Total forces' in Table XXVII

$$\left.\begin{aligned} F_1 = 0{\cdot}28, \qquad F_2 = -0{\cdot}01, \qquad F_3 = 0{\cdot}73866667, \\ F_4 = -0{\cdot}63757576,\ F_5 = 0{\cdot}82424242,\ F_6 = -0{\cdot}68933333, \end{aligned}\right\} \quad (34)$$

and these values (multiplied by 1,000 for convenience) we insert as 'initial forces' in the first line of a relaxation table.

153. The relaxation table (in which operations were performed by slide-rule) is not reproduced but is summarized in Table XXIX: there, multiplications have been carried to eight significant figures with the aid of a calculating machine, and all forces (for convenience) are multiplied by 1,000. It will be noticed that account is kept (in columns to the right) of forces of the type T_k: this is with a view to periodical improvement of the approximation of λ, with guidance obtained from upper and lower limits calculated in accordance with (25).

That rule implies that an assumed value p^2 will be too high if at any stage of the relaxation process, as in line 9 of Table XXIX, all such quantities as $a_k F_k$ and $a_k T_k$ are positive. In lines 8–10, computations which are self-explanatory in the light of (25) show that an addition $\Delta\lambda$ of $-0{\cdot}00413139$ will leave λ still too high. That is to say, the required value of λ,

$$\lambda_1 \quad \text{(say)} \quad < 0{\cdot}02586861. \tag{35}$$

Accordingly in the second stage of relaxation we investigate the effect of a somewhat larger alteration (namely, $\Delta\lambda = -0{\cdot}00415$), taking as our next trial value $\lambda = 0{\cdot}02585$.

154. Line 13 of Table XXIX gives the residual forces as altered by this change in the value of λ. As was to be expected, some forces have their signs reversed, others remain positive. The new values are

$$38{\cdot}35,\ -25{\cdot}346,\ -4{\cdot}35853,\ -3{\cdot}60401,\ -6{\cdot}14624,\ 13{\cdot}15813, \tag{36}$$

correctly to the fifth decimal place. (As in § 153, a calculating machine was used to obtain them with this approximation.)

Lines 14–18 of Table XXIX summarize the second stage of the relaxation process, made with an operations table similar to Table XXVIII but differing on account of the altered value of λ. This stage was stopped when all quantities of the type $a_k F_k$ had become negative (line 19), so that (all quantities of type $a_k T_k$ being positive) according

to (25) a *positive* addition $\Delta\lambda$ was required. Calculations which are reproduced in lines 20–22 gave the required addition as

$$\Delta\lambda = 10{\cdot}794 \times 10^{-6},$$

thus showing that (35) can be replaced by

$$\lambda_1 < (0{\cdot}02585 + 10{\cdot}793 \times 10^{-6}) = 0{\cdot}025860794. \qquad (37)$$

155. The ratio F_k/T_k has the following values at the end of the first stage (line 7) and of the second stage (line 19) respectively:

$k =$		1	2	3	4	5	6
Line 7:	$10^{-2}\times$	42·5	0·649	2·063	2·970	3·879	8·411
Line 19:	$10^{-6}\times$	−62·5	−54·935	−53·499	−5·224	−1·290	−11·640

Therefore in the notation of § 145

$$\lambda - \left(\frac{F}{T}\right)_L = 0{\cdot}03 - 0{\cdot}0425, \text{ at the close of the first stage,}$$

$$= 0{\cdot}02585 - (-1{\cdot}29 \times 10^{-6}), \text{ at the close of the second stage.}$$

Combining these results with (35) and (37), we have finally, as 'brackets' for λ_1,

$$-0{\cdot}0125 < \lambda_1 < 0{\cdot}02586861 \qquad (38)$$

at the end of the first stage, and

$$0{\cdot}02585129 < \lambda_1 < 0{\cdot}025860794 \qquad (39)$$

at the end of the second.

Since λ_1 is known *ab initio* to be positive, the lower limit in (38) is nugatory; but applied to the figures resulting from the second stage in Table XXIX our rule has yielded a very close 'bracket'. The total range of uncertainty is $9{\cdot}5 \times 10^{-6}$, so that by giving to λ the mean value

$$\tfrac{1}{2}(0{\cdot}02586079 + 0{\cdot}02585129) = 0{\cdot}02585604$$

we shall certainly keep the error within

$$\frac{4{\cdot}75}{25{,}856} = 0{\cdot}018_4 \text{ per cent.} \qquad (40)$$

This result is verified in § 149 of Chapter VII.

General theory (continued): Determination of higher modes and frequencies

156. When as in this example λ_1 (i.e. p_1^2) has been bracketed between limits sufficiently close, calculation can be stopped and we are left with close estimates not only of p_1^2 (to which an approxima-

tion could have been obtained by other methods based on Rayleigh's principle) but also of the associated mode (which that principle does not determine). On this account, making use of the conjugate property which is a characteristic of 'normal modes', we can proceed to similar calculations of p_2^2, p_3^2,..., etc. and of the modes associated with those higher frequencies. This is an important merit of the relaxation procedure.

The conjugate or 'orthogonal' property of normal modes (§ 133 and *Elasticity* § 504) implies that equation (4) can be written in the form of (12),—$\mathbf{V}_1$, $\mathbf{V}_2$,..., $\mathbf{V}_N$ and $\mathbf{T}_1$, $\mathbf{T}_2$,..., $\mathbf{T}_N$ denoting the values of $\mathbf{V}$ and $\mathbf{T}$ for the first, second,..., Nth normal modes,—if the mode is regarded as composed of these N modes combined in the proportion $\phi_1 : \phi_2 : \ldots : \phi_N$. It follows that

$$\frac{\partial p^2}{\partial \phi_r} = 0, \quad \text{when all the } \phi\text{'s are zero except } \phi_r,$$

so that (in accordance with § 131) p^2 as given by (4) has a stationary value when the mode is normal. But (12) shows further that if ϕ_1 is zero then the smallest of such stationary values is p_2^2: that is to say, *p_2^2 is the minimum value of p^2 as given by* (4) *provided that the mode has no 'first normal component'*.

157. When an assumed mode satisfies this requirement, in normal coordinates $\mathbf{V}$ and $\mathbf{T}$ will have the forms

$$\mathbf{V} = \phi_2^2 \mathbf{V}_2 + \phi_3^2 \mathbf{V}_3 + \ldots, \qquad \mathbf{T} = \phi_2^2 \mathbf{T}_2 + \phi_3^2 \mathbf{T}_3 + \ldots,$$

and so, for the assumed mode,

$$\frac{\partial \mathbf{V}}{\partial \phi_1} = \frac{\partial \mathbf{T}}{\partial \phi_1} = 0$$

whatever be the magnitudes of ϕ_2, ϕ_3,..., etc. This means that no work *on the whole* will be done by either the elastic or kinetic forces of the assumed mode in acting through displacements corresponding with the first normal mode (ϕ_1); and the Reciprocal Theorem (*Elasticity* § 12) shows that in consequence *no work would be done by either the elastic or kinetic forces of the first mode in acting through the displacements of the assumed mode.* We thus obtain a relation which must hold between the displacements in an assumed mode in order that this shall have no 'first normal component'; and to modes thus restricted Rayleigh's principle (§ 131) can be applied with the modification that p_2^2 is now the minimum value which can be assumed by p^2 as calculated from (4). Clearly the argument can be extended.

158. To formulate the required relation, we denote by $(V_k)_1$, $(T_k)_1$ the typical elastic and kinetic force in the first mode. Then, if a_k typifies the displacement in the assumed mode, the above criterion requires that

$$\left.\begin{aligned} a_1(V_1)_1+a_2(V_2)_1+\ldots+a_N(V_N)_1 &= 0, \\ a_1(T_1)_1+a_2(T_2)_1+\ldots+a_N(T_N)_1 &= 0, \end{aligned}\right\} \tag{41}$$

these relations being equivalent because the ratio T_k/V_k has the same value, by (5), for all values of k. Knowing the first mode we can give numerical values to $(V_1)_1,\ldots$, etc.

Since it demands that (41) be satisfied, the requirement that the assumed mode shall have no 'first component' in effect reduces the order of freedom by 1; but when the requirement is satisfied, then p^2 as given by (4) can in no case be less than p_2^2. Having determined p_2^2 and its associated mode we can similarly calculate p_3^2, and so on.

Numerical example (continued): Second mode and frequency

159. We now apply these methods to the torsional system of § 147, Chapter VII. Here the forces $(T_1)_1,\ldots, (T_N)_1$ have values given in line 19 of Table XXIX; consequently the second of equations (41) is

$$1+7\cdot117080a_2+2\cdot055192a_3+3\cdot010880a_4+ \\ +22\cdot412280a_5+3\cdot064384a_6 = 0, \tag{42}$$

on the assumption made previously (§§ 148–9) that a_1 is unity. We know that the second mode must have a node at some point in the shaft; so a suitable form is obtained by multiplying the displacements $a_1, a_2,\ldots, a_6$ of the first mode by $1, 1-\alpha, 1-2\alpha,\ldots, 1-5\alpha$ respectively, and then choosing α so that (42) is satisfied.

The resulting value of α is $0\cdot2780847_0$, and the corresponding displacements are

$$\begin{gathered} a_1 = 1,\quad a_2 = 1\cdot284482_2,\quad a_3 = 1\cdot140196_4,\quad a_4 = 0\cdot499041_0, \\ a_5 = -0\cdot419628_1,\qquad a_6 = -1\cdot495509_4. \end{gathered} \tag{43}$$

Table XXX (pp. 156–7) is constructed for these displacements in the manner of Table XXVII. It gives for λ a value 0·3016, and thus suggests the assumption

$$\lambda = 0\cdot3 \tag{44}$$

for the first stage of the relaxation process. The corresponding initial forces are obtained in the last three lines.

160. Again the relaxation table is not reproduced, and here the tabular summary (Table XXXI) is condensed still further. Operations effected by slide-rule, and without regard to (42),† suggested additions to the displacements which were then modified (with the use of a machine) so that (42) *was* satisfied; finally their consequences were calculated (again by machine) and recorded in Table XXXI. The modification consisted in an addition of 'first-mode displacements' as recorded in line 20 of Table XXIX, multiplied by a constant determined from (42). By this means advantage was taken of the calculations in lines 19–22 of the same table: thus the additions to (43) suggested by a rough liquidation were

$$\left.\begin{array}{c}\Delta a_1 = 0, \qquad \Delta a_2 = 0{\cdot}3, \qquad \Delta a_3 = -0{\cdot}2, \qquad \Delta a_4 = -0{\cdot}07, \\ \Delta a_5 = -0{\cdot}2, \qquad \Delta a_6 = 0{\cdot}7,\end{array}\right\} \quad (45)$$

and accordingly β, the multiplying constant, could be deduced from the equation

$$123{\cdot}464817\beta+0{\cdot}3\times 7{\cdot}117080-0{\cdot}2\times 2{\cdot}055192-$$
$$-0{\cdot}07\times 3{\cdot}01088-0{\cdot}2\times 22{\cdot}41228+0{\cdot}7\times 3{\cdot}064384 = 0.$$

The result was $\qquad \beta = 0{\cdot}006674478,$

and in consequence the corrections (45) were altered to

$$\left.\begin{array}{lll}\Delta a_1 = 0{\cdot}006674, & \Delta a_2 = 0{\cdot}311876, & \Delta a_3 = -0{\cdot}182853, \\ \Delta a_4 = -0{\cdot}049904, & \Delta a_5 = -0{\cdot}175068, & \Delta a_6 = 0{\cdot}725566,\end{array}\right\} \quad (46)$$

which satisfy (42) and entail the forces shown in line 2 of Table XXXI. It will be observed (line 4) that a_1 is no longer exactly unity: nevertheless by keeping it unity in the trial liquidation we have attained our object of avoiding a null solution.

Table XXXI is self-explanatory, except as regards the corrections applied in the different stages to the displacements. In the first stage (line 2) these are given in (46); in the second stage (line 10) they are

$$\left.\begin{array}{lll}\Delta a_1 = 0{\cdot}000480, & \Delta a_2 = 0{\cdot}030853, & \Delta a_3 = -0{\cdot}007768, \\ \Delta a_4 = -0{\cdot}008556, & \Delta a_5 = -0{\cdot}008208, & \Delta a_6 = 0{\cdot}001837;\end{array}\right\} \quad (47)$$

† By waiving this requirement we greatly simplify the individual operations. In some instances lower limits were computed before the lower modes had been eliminated.

TABLE XXX

$k =$	T_k					
	1	2	3	4	5	6
$a_1 = 1$	1	..	..	..	..	..
$a_2 = 1{\cdot}284482$	..	5·137929	..	..	..	..
$a_3 = 1{\cdot}140196$	..	..	0·912157	..	..	..
$a_4 = 0{\cdot}499041$	..	..	..	0·499041	..	..
$a_5 = -0{\cdot}419628$	..	..	..	..	−2·517769	..
$a_6 = -1{\cdot}495509$	..	..	..	..	..	−1·196407
Total forces	1	5·137929	0·912157	0·499041	−2·517769	−1·196407
a_k	1	1·284482	1·140196	0·499041	−0·419628	−1·495509
Products	1	6·599577	1·040038	0·249042	1·056526	1·789239
	Total = 2T = 11·734422			(λ = V/T = 0·3016)		
$0{\cdot}3T_k$	0·3	1·541379	0·273647	0·149712	−0·755331	−0·358922
$-V_k$	−0·644397	−0·499888	−0·924306	0·233439	−0·061414	+0·896568
Sum = $\mathbf{F}_k$	−0·344397	1·041490	−0·650659	0·383151	−0·816745	0·537645

and in the third stage (line 18) they are

$$\left.\begin{aligned} \Delta a_1 &= -0{\cdot}003140, & \Delta a_2 &= 0{\cdot}000107, & \Delta a_3 &= -0{\cdot}000445, \\ \Delta a_4 &= -0{\cdot}000818, & \Delta a_5 &= 0{\cdot}000225, & \Delta a_6 &= 0{\cdot}000231. \end{aligned}\right\} \quad (48)$$

Equation (42) is satisfied both by (47) and (48).

The progress of the approximation is indicated by the narrowing inequalities, or 'brackets', which follow lines 6, 14, and 22. According to the last of these, giving to λ the mean value 0·2239265 we can be sure that the error is within

$$\frac{4{\cdot}31}{22{,}388} = 0{\cdot}019_{25} \text{ per cent.,} \quad (49)$$

and the result is verified in § 149 of Chapter VII.

Higher modes and frequencies

161. The procedure for determining the third and higher modes and frequencies is exactly similar. The third mode must be 'orthogonal' not only to the first but also to the second: therefore its displacements must satisfy not only (42) but also the equation

$$1{\cdot}004014a_1 + 6{\cdot}509274a_2 + 0{\cdot}759304a_3 + 0{\cdot}439763a_4 - \\ -3{\cdot}616078a_5 - 0{\cdot}614300a_6 = 0, \quad (50)$$

which can be based on an exactly similar argument. To obtain a starting assumption we multiply the displacements $a_1, a_2,..., a_6$ in the second mode by $1, 1-\gamma, 1-2\gamma,..., 1-5\gamma$ respectively, then choose

TABLE XXX (*cont.*)

2·25	−1·25	..	..	..	..
−1·605603	2·890085	−1·284482	..	..	..
..	−1·140196	3·040523	−1·900327	..	..
..	..	−0·831735	1·285409	−0·453674	..
..	..	..	0·381480	−0·731170	0·349690
..	..	..	..	1·246258	−1·246258
0·644397	0·499889	0·924306	−0·233439	0·061414	−0·896568
1	1·284482	1·140196	0·499041	−0·419628	−1·495509
0·644397	0·642098	1·053891	−0·116495	−0·025771	+1·340825

Total = 2V = 3·538945 ($\lambda = \mathbf{V}/\mathbf{T} = 0{\cdot}3016$)

γ so that (50) is satisfied. In this way we obtain a mode having one more nodal point and containing no 'second normal component': if now we correct its displacements in the manner of § 159, what results has neither a first nor a second normal component.

In the liquidation process we start as before (§ 160) by slide-rule and without satisfying (42) and (50), then with the use of a machine modify the suggested additions and record their consequences. Now, of course, a double correction is necessary: to eliminate the first normal component from a set of displacements $a_1, a_2,..., a_6$ we must add first-mode displacements multiplied by β_1, where

$$123{\cdot}464817\beta_1+a_1+7{\cdot}117080a_2+2{\cdot}055192a_3+3{\cdot}010880a_4+ \\ +22{\cdot}412280a_5+3{\cdot}064384a_6 = 0, \quad (51)$$

and to eliminate the second normal component we must add second-mode displacements multiplied by β_2, where

$$15{\cdot}165819\beta_2+1{\cdot}004014a_1+6{\cdot}509274a_2+0{\cdot}759304a_3+ \\ +0{\cdot}439763a_4-3{\cdot}616078a_5-0{\cdot}614300a_6 = 0. \quad (52)$$

162. Every new mode and frequency (except the sixth, which results *at once* when all graver components have been eliminated)† entails a separate relaxation process and a summary of the type of Tables XXIX and XXXI, so six tables of this kind were entailed by the problem of Fig. 33. The last four are not reproduced, but are

† In this last instance, of course, Rayleigh's principle provides the *lower* limit, and an upper limit can be found in the manner of § 145 from (24) with $(\mathbf{F}/T)_L$ replaced by $(\mathbf{F}/T)_S$, the *smallest* value of $\mathbf{F}/T$. Improvement was not found to be practicable: thus the narrow range of uncertainty as regards λ_6 in (30) is an indication of the accuracy with which the lower modes had been computed.

briefly summarized in Tables XXXII–XXXV to show how the solutions converge. The results are verified in § 149 of Chapter VII.

Torsional systems which are entirely unrestrained

163. We have thus determined within narrow limits of error all of the natural frequencies, together with the associated modes, of the torsional system specified in § 147 of Chapter VII. That system is in one respect artificial, namely, in that the end point O of Fig. 33 is fixed and therefore nodal for every mode: torsional systems usually terminate at each end with a finite mass which takes part in the vibration,—there is no constraint entailing an external force, therefore every normal vibration must conform with the principle of the Conservation of Moment of Momentum. That is to say, in every vibration the total moment of momentum $\sum[I_k.\dot{q}_k]$ must have a constant value throughout all time, and so in a normal vibration, according to (2),

$$\sum[I_k.a_k] = 0. \tag{53}$$

Now (53), as implying a linear relation between the a's, will affect the computations exactly as in §§ 159–60 they were affected by (42), the condition for no 'first normal component'. Like (42), in effect it reduces the order of the freedom by 1, and makes the gravest frequency take a higher value than it has when (one end being fixed) all masses can move together in the same direction. We can if we like regard the fixity of O in Fig. 33 as due to the attachment there of an infinite mass: then on account of this added mass N is increased from 6 to 7, but it is again reduced to 6 (as in our calculations) when we impose the momentum condition (53).†

RECAPITULATION

164. This chapter is in direct continuation of Chapter VII, with which its equation numbers run consecutively. It has been taken almost in its entirety from the two papers cited below.

Chapter VII provided foundations for a relaxation treatment of 'characteristic number problems', in the form of two theorems (both due to Lord Rayleigh) whereby upper and lower limits can be imposed on a wanted value λ. Here the relaxation treatment is described. A mode is assumed and used to calculate a trial value of λ which (by 'Rayleigh's principle') will not be very wide of the mark. The error is presented as a set of 'residual forces' calling for

† On this aspect, cf. Ref. 2, Part II.

TABLES XXXII–XXXV. *Summarized Results for the Higher Modes and Frequencies*

Limits imposed on λ		*Displacements in the corresponding mode*					
		a_1	a_2	a_3	a_4	a_5	a_6
Table XXXII: 3rd mode							
1st stage	$0{\cdot}998332 < \lambda_3 < 1{\cdot}159069$	+0·338890	+0·203307	−0·851204	−1·008288	+0·031210	+0·750523
2nd stage	$1{\cdot}102694 < \lambda_3 < 1{\cdot}112744$	+0·258677	+0·232939	−0·841638	−1·033750	−0·109284	+1·754027
3rd stage	$1{\cdot}111714 < \lambda_3 < 1{\cdot}112426$	+0·258239	+0·235217	−0·840769	−1·036127	−0·127633	+1·884832
4th stage	$1{\cdot}112404 < \lambda_3 < 1{\cdot}1124246$	+0·258319	+0·235147	−0·839604	−1·036192	−0·128461	+1·890370
Table XXXIII: 4th mode							
1st stage	$0{\cdot}907112 < \lambda_4 < 1{\cdot}326799$	+0·198410	+0·213154	−0·903893	−1·098391	+0·312674	−1·161203
2nd stage	$1{\cdot}320314 < \lambda_4 < 1{\cdot}320656$	+0·262771	+0·195087	−0·919453	−1·005190	+0·297671	−1·111658
3rd stage	$1{\cdot}320533 < \lambda_4 < 1{\cdot}3206425$	+0·262352	+0·195103	−0·919200	−1·005067	+0·297614	−1·111429
Table XXXIV: 5th mode							
1st stage	$-3{\cdot}155846 < \lambda_5 < 2{\cdot}673405$	+1·068596	−0·194316	+0·313160	−0·106366	−0·000956	+0·004064
2nd stage	$2{\cdot}410248 < \lambda_5 < 2{\cdot}458006$	+1·347033	−0·224312	+0·018600	+0·138864	−0·010271	+0·007597
3rd stage	$2{\cdot}457708 < \lambda_5 < 2{\cdot}457992$	+1·346877	−0·224125	+0·015654	−0·140984	−0·010341	+0·007615
Table XXXV: 6th mode							
	$4{\cdot}912809 < \lambda_6 < 4{\cdot}916035$	+0·057970	−0·123640	+2·078960	−1·501980	+0·049640	−0·013360

liquidation, and a relaxation process is started in the usual manner. When this has gone some way, the basic theorems are employed to calculate upper and lower limits between which λ must lie, and the whole process can then be repeated with an estimated value nearer to the truth than before. As liquidation continues the two limits steadily converge: hence λ *and its associated mode* can be estimated with any accuracy that may be desired.

165. Here, in the computations, more significant figures are retained than the accuracy of the physical data can justify, because the aim is to impose a stringent test on *methods*. It is very seldom that engineering data justify more figures than can be read from a 20-inch slide-rule: in this chapter (cf. § 152) eight significant figures, in general, have been retained by recourse to calculating machines.

A few remarks may be made in this connexion. First, there can be no doubt of the value of these machines to the 'relaxational' computer. It is the fact that results of more than three-figure accuracy will have no meaning in practical work: but liquidation is an additive process, and a long series of operations will result, almost inevitably, in accumulated errors; so time on the whole will be saved by retaining decimals that have no physical meaning, thus relegating the errors to figures which can be rejected *at the finish* as not significant. Considering the nature of the relaxation process, computers are advised not to take thought about 'dishonest decimals' until their final results have to be presented.

On the other hand machines should not be used unthinkingly, as though time-saving were an inevitable result: this is to waste an advantage of the relaxation process, that in effect (§ 18) after every operation it starts afresh. Initially, when the 'forces on constraints' are large, progress will be much faster if the aim is only approximate liquidation, performed with the aid of a slide-rule; and although the effects of the calculated displacements will not (by reason of accumulated errors) be exactly what have been given by slide-rule, they will on the other hand leave residuals much smaller than the original forces. It is at this point that a machine is really valuable: *accepting the displacements as calculated by slide-rule, we can use it to find their true effects*. Then, at the finish, we have forces considerably reduced and still given with precision; so liquidation can continue, and again (in the first instance) a slide-rule can be employed.

These remarks are well exemplified by Table XXXI of the present chapter. The displacement corrections in line 2 were based on values (45) derived by a rough (slide-rule) liquidation: the consequent forces would be given by slide-rule only to three or four places, but a machine gave the accurate values which are stated in line 2, consequently the residual forces of line 3 are known with certainty. Lines 4–8 similarly required the use of a machine, in order that the 'corrected residuals' of line 9 might be given reliably: the largest of these is less than 5 per cent. of the largest initial force (line 1), so a second 'slide-rule liquidation' could be used to bring them all below 10,000 (line 11), and in line 19 a similar liquidation has brought them all below 1,000. Thus a slide-rule used for liquidation has disposed of initial forces within less than 0·1 per cent. (an accuracy quite outside its own range), because on three occasions its conclusions have been corrected by machine.

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IX

VIBRATING SYSTEMS IN GENERAL. FORCED OSCILLATIONS, AND THE TREATMENT OF DISSIPATIVE FORCES

166. SPACE will not permit a similarly detailed treatment of other problems relating to undamped free vibrations; but a careful study of Chapters VII and VIII (and more particularly of § 147) should serve to convince the reader that the one example there treated is in fact sufficient,—its torsional system exhibits every feature which can characterize a system devoid of friction. Regarded as functions of six independent variables the expressions (27) for **V** and **T** are more than usually concise, and in consequence more than is usual of the entries in Table XXVI are zeros; but that fact makes little difference to the time and labour of the liquidation process. Because its basis is the Lagrangian equations (§ 129), which hold without restriction on the coordinates, within its range the whole of our treatment is completely general. *Any* significance can be attached to the 'generalized coordinates' a_1, a_2,..., a_N, and except in the formulation of **V** and **T** no geometrical or physical thinking is demanded.

As yet, however, we have not contemplated either forces applied to a system from outside or damping forces intrinsic in the system itself, and so, though general, our treatment is not complete. But even when these additional factors call for inclusion it forms, as we now proceed to show, a convenient first stage in the fuller treatment. That is to say, knowing the modes and frequencies of undamped free vibration we are better equipped for a discussion of forced vibrations and of dissipative or frictional forces.

Forced oscillations in systems devoid of damping

167. Non-oscillatory forces call for very little discussion. The form of the complete Lagrange equations, viz.†

$$\frac{\partial(\mathfrak{V}-\mathfrak{T})}{\partial q_k}+\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{q}_k}\right) = Q_k \quad (k = 1, 2, 3, \ldots, N) \qquad (1)$$

shows that any number of distinct solutions can be superposed, and evidently steady forces will entail steady displacements. Now

† Cf. § 129 of Chapter VII.

when all the q's are steady $\mathfrak{T}$ will disappear from (1), which thus reduce to

$$\frac{\partial \mathfrak{V}}{\partial q_k} = Q_k \quad (k = 1, 2, 3, ..., N) \tag{2}$$

and can be solved in the manner of Chapters I–III. Then we shall have the configuration of statical equilibrium, and this can be made the datum configuration relatively to which $\mathfrak{V}$ is calculated (§ 127). Disposing in this way of steady forces, we shall be left with equations of vibration to be solved exactly as before.

168. In special circumstances (i.e. of 'resonance') large oscillations may result from fluctuating forces even when these are small: nevertheless we shall develop a theory of forced oscillations on the assumption made previously (§ 129) that $\mathfrak{T}$ is a quadratic function of the $\dot{q}$'s having constant coefficients, and $\mathfrak{V}$ a similar function of the q's. Then, as in § 129, quantities of the type $\partial\mathfrak{T}/\partial q_k$ will vanish; and if all the Q's vary with time in accordance with a common time-factor $\sin(pt+\epsilon)$, then assuming that

$$\left.\begin{aligned} Q_k &= \mathsf{A}_k \sin(pt+\epsilon) \\ q_k &= a_k \sin(pt+\epsilon) \end{aligned}\right\} \quad (k = 1, 2, 3, ..., N) \tag{3}$$

(i.e. that all displacements fluctuate with the forces both as regards frequency and phase), we can reduce the Lagrangian equations to the simpler forms

$$\frac{\partial \mathbf{V}}{\partial a_k} - p^2 \frac{\partial \mathbf{T}}{\partial a_k} = \mathsf{A}_k \quad (k = 1, 2, 3, ..., N), \tag{4}$$

in which $\mathbf{V}$ and $\mathbf{T}$ have the same significance as in Chapter VII. These are the general equations of forced vibration.

There will be N equations (4), and N quantities to be determined, —namely, $a_1, a_2, ..., a_N$. The A's are specified, so now the number of equations is exactly what is needed, and the solution will be *unique*. Thus our assumption regarding the Q's is justified: that is to say, all Q's which have a common time-factor may be grouped into a single system and the corresponding displacements calculated; then the different solutions may be combined in accordance with the principle of superposition.

169. Two lines of attack on (4) may be developed, according as a solution is required for some definite frequency of the applied forces or to cover a range of frequencies. (The second problem arises when

we want to investigate the frequencies for which forced vibrations may be dangerously large.)

In either event, if the problem is attacked by Relaxation Methods, the forces which call for liquidation will be given not by (15) of Chapter VII but by such expressions as

$$\mathbf{F}_k = \mathsf{A}_k + F_k = \mathsf{A}_k + p^2 T_k - V_k, \tag{5}$$

in which A_k the external force and p^2 (defining its frequency of pulsation) are to be regarded as data, and the procedure required for liquidation will apply without change. When p^2 is specified *numerically*, only one 'stage' will be entailed.

170. When on the other hand the response is wanted for more than one periodicity of the applied forces, a solution for free vibrations such as we have obtained in Chapter VIII provides all of the material required for a solution. If $(T_1)_1$, $(T_2)_1$,..., $(T_N)_1$ stand as in § 158 for the inertia forces in the first mode, and if p_1^2 relates to the natural frequency in that mode, then a first-mode displacement associated with some other frequency-parameter p^2 will entail additions to the forces on constraints of which

$$F_k = p^2(T_k)_1 - (V_k)_1$$

is typical: therefore according to (5) all forces on constraints will vanish provided that for every k the external force

$$\begin{aligned} \mathsf{A}_k = -F_k &= (V_k)_1 - p^2(T_k)_1 \\ &= (p_1^2 - p^2)(T_k)_1, \end{aligned}$$

since $\qquad V_k = p_1^2 T_k \quad (k = 1, 2, ..., N)$

in the first-mode displacement. This means that if for all k's the external force

$$\mathsf{A}_k = A\,.\,(T_k)_1 \tag{6}$$

(A having any constant value), the response of the system will consist of first-mode displacements multiplied in the ratio

$$A/(p_1^2 - p^2), \tag{7}$$

—a well-known result. Similar relations hold in relation to the other normal modes: so we can calculate the response to any given system of external forces, pulsating with any periodicity, provided that we can express every external force in a series of the form

$$\mathsf{A}_k = A_1(T_k)_1 + A_2(T_k)_2 + ... + A_N(T_k)_N \tag{8}$$

in which the A's have values independent of k.

Example 1: Forced oscillations of an undamped torsional system

171. Taking advantage of the conjugate property (§ 132) it is easy to evaluate the A's in (8). For in virtue of that property we have

$$\sum_N [\mathsf{A}_k.(a_k)_r] = A_r \sum_N [(T_k)_r(a_k)_r] = 2A_r.\mathbf{T}_r \tag{9}$$

simply, $(a_k)_r$ denoting the displacement a_k, and $\mathbf{T}_r$ the value of $\mathbf{T}$, appropriate to the rth normal mode. Thus suppose that in the torsional example of Fig. 33 (p. 145) a unit 'forcing couple' is applied to the fifth mass (I_5), so that

$$\mathsf{A}_1 = \mathsf{A}_2 = \mathsf{A}_3 = \mathsf{A}_4 = \mathsf{A}_6 = 0, \qquad \mathsf{A}_5 = 1, \tag{10}$$

and that we want A_2. The displacements in the second mode are given in line 20, and the corresponding T-forces in line 19 of Table XXXI; moreover in line 22 of that table the value of $\sum_N [(T_k)_2.(a_k)_2]$, i.e. of $2\mathbf{T}_r \times 10^6$, is recorded as 15,165,819. Therefore in this instance (9) becomes

$$15{\cdot}165819 A_2 = 1 \times (-0{\cdot}602680). \tag{11}$$

The other A's can be calculated similarly, and solutions obtained in this way for other forcing couples acting severally can be combined in accordance with the Principle of Superposition.

Approximate treatment of systems which include damping forces

172. It should be emphasized that this general treatment of forced oscillations is possible only for the reason that using relaxation methods we can determine the modes as well as the frequencies of free vibration. A further consequence is the possibility of making allowance for damping forces *provided that these are small.*

When (as is usual) the damping forces can be derived from a 'dissipation function' $\mathfrak{F}$ which is a homogeneous quadratic function of the generalized velocities,† equation (1) is replaced by

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{q}_k}\right) - \frac{\partial \mathfrak{T}}{\partial q_k} + \frac{\partial \mathfrak{F}}{\partial \dot{q}_k} + \frac{\partial \mathfrak{V}}{\partial q_k} = Q_k, \tag{12}$$

and if $\mathfrak{F}$ is necessarily positive then all free vibrations will come ultimately to zero. Consequently if normal free vibrations (of

† Cf. (e.g.) Ref. 2, § 96. The dissipation function was first introduced by Rayleigh (Ref. 4): $2\mathfrak{F}$ measures the rate at which the total energy is (at any particular instant) being diminished by friction.

small amplitude) be now defined by the statement that every coordinate varies in accordance with a common time-factor $e^{\lambda t}$, so that

$$q_k = a_k e^{\lambda t} \quad (a_k \text{ constant}), \tag{13}$$

and if $\mathfrak{T} = \lambda^2 \mathbf{T} e^{2\lambda t}, \quad \mathfrak{F} = \lambda^2 \mathbf{\Phi} e^{2\lambda t}, \quad \mathfrak{V} = \mathbf{V} e^{2\lambda t},$ (14)

so that each of $\mathbf{T}$, $\mathbf{\Phi}$, $\mathbf{V}$ is a homogeneous quadratic function of the N displacements typified by a_k, then λ will either be real and negative or complex with negative real part. The type equation for an undamped system, viz.

$$V_k - p^2 T_k = 0,$$

is replaced by

$$\lambda^2 T_k + \lambda \Phi_k + V_k = 0, \tag{15}$$

in which Φ_k (conformably with our previous notation) stands for $\partial \mathbf{\Phi}/\partial a_k$; and because λ may be complex, in a solution of the N equations typified by (15) we must contemplate the possibility that some or all of the N displacements of type a_k—i.e. of the $(N-1)$ ratios of type a_k/a_1—will also be complex.

173. Formally the orthodox treatment of normal vibrations is not greatly altered by the inclusion of $\mathbf{\Phi}$, but in practice the labour becomes prohibitive when N exceeds 3 or 4, since the order of the 'frequency equation' is now raised from N to $2N$. In ordinary applications the damping forces are small, and on that understanding it was shown by Rayleigh (Ref. 5, § 102) that approximate solutions can be found. The main effects of damping are revealed as (i) the imposition on each normal oscillation of some finite decay factor, unaccompanied by change in periodicity, and (ii) a modification of each mode whereby the different coordinates no longer fluctuate in phase. For systems characterized by heavy damping, or by damping which has opposite signs in different parts, no practicable alternative to the orthodox treatment appears to have been devised.

174. An approximate treatment of *small and positive* damping can be based on the method of Chapter VIII. When by that method all of the N normal modes and associated frequencies of a system have been determined, we can change from the coordinates $a_1, a_2, \ldots, a_N$ to 'normal coordinates' $\phi_1, \phi_2, \ldots, \phi_N$ which are the multiplying constants required to make

$$a_k = \phi_1 (a_k)_1 + \phi_2 (a_k)_2 + \ldots + \phi_N (a_k)_N \tag{16}$$

hold for all k's, $(a_k)_r$ having the same significance as in § 171. Then the expressions for $\mathbf{T}$ and $\mathbf{V}$ can be replaced by†

$$\left.\begin{aligned} \mathbf{T} &= \phi_1^2\mathbf{T}_1+\phi_2^2\mathbf{T}_2+...+\phi_N^2\mathbf{T}_N, \\ \mathbf{V} &= \phi_1^2\mathbf{V}_1+\phi_2^2\mathbf{V}_2+...+\phi_N^2\mathbf{V}_N, \\ \text{where} \quad \mathbf{V}_r &= p_r^2.\mathbf{T}_r = \mu_r\mathbf{T}_r, \quad \text{say,} \end{aligned}\right\} \tag{17}$$

all μ's being known from our solution. $\mathbf{T}_r$ and $\mathbf{V}_r$ (as in § 133) denote the (known) values of $\mathbf{T}$ and $\mathbf{V}$ appropriate to the rth normal mode.

In general the corresponding expression for $\mathbf{\Phi}$ will involve products as well as the squares of the normal coordinates, but it will always be a homogeneous quadratic function: therefore in (15), *which will also hold when T_k, Φ_k, V_k are interpreted as standing for $\partial\mathbf{T}/\partial\phi_k$, $\partial\mathbf{\Phi}/\partial\phi_k$,* $\partial\mathbf{V}/\partial\phi_k$, Φ_k will be a linear function of (in general) all the ϕ's. Using (17) as expressions for $\mathbf{T}$ and $\mathbf{V}$, we can derive N equations typified by

$$2\mathbf{T}_k(\lambda^2+\mu_k)\phi_k+\lambda\Phi_k = 0, \tag{18}$$

which when $\mathbf{\Phi} = 0$ yield N solutions typified by

$$\lambda^2+\mu_k = 0, \qquad \phi_k \text{ unrestricted}, \qquad \phi_r = 0 \quad (r \neq k), \tag{19}$$

—as we should expect. Starting with any one of these solutions we now trace the effect of $\mathbf{\Phi}$ *assumed finite but small.*

175. When the starting solution is that in which ϕ_k does not vanish, it follows from (19) that all ϕ's except ϕ_k (since they vanish with $\mathbf{\Phi}$) may for a first approximation be neglected in the expression for Φ_k. Accordingly in (18) we may write

$$\left.\begin{aligned} &\Phi_k \approx \alpha_{kk}.\phi_k, \\ \text{where} \quad &\alpha_{kk} \equiv \frac{\partial^2\mathbf{\Phi}}{\partial\phi_k^2}, \quad \text{so is small of order } \mathbf{\Phi}. \end{aligned}\right\} \tag{20}$$

Then (18) reduces to

$$\{2\mathbf{T}_k(\lambda^2+\mu_k)+\lambda\alpha_{kk}\}\phi_k = 0, \tag{21}$$

and for $r \neq k$ we have to a like approximation

$$\left.\begin{aligned} &2\mathbf{T}_r(\lambda^2+\mu_r)\phi_r+\lambda\alpha_{kr}\phi_k = 0, \\ \text{where} \quad &\alpha_{kr} \equiv \frac{\partial^2\mathbf{\Phi}}{\partial\phi_k\partial\phi_r}, \quad \text{so is small of order } \mathbf{\Phi}. \end{aligned}\right\} \tag{22}$$

If now we postulate that $\phi_k = 1$ in the kth mode as modified by

† The ϕ's in this section have the same significance as in § 133, and equations (17) are compatible with (12) of that section.

damping,† then the solution of (21) is (to the approximation of this section)

$$\left.\begin{aligned} \lambda = \lambda_k = -\frac{1}{4}\frac{\alpha_{kk}}{\mathrm{T}_k} \pm i\sqrt{\mu_k}, \\ \text{and to a like approximation we have from (22)} \\ (\mu_k-\mu_r)\phi_r = \pm\frac{i\alpha_{kr}\sqrt{\mu_k}}{2\mathrm{T}_r} \quad (r \neq k). \end{aligned}\right\} \tag{23}$$

176. To express this solution in real form we recall that the a's and ϕ's are related by linear equations of the type of (16). Hence, according to the starting assumption which was expressed in (13), the q's and ϕ's are related by corresponding equations of the type

$$q_r = e^{\lambda t}\{\phi_1(a_r)_1+\phi(a_r)_2+\ldots+\phi_N(a_r)_N\}. \tag{24}$$

Now the q's are governed by Lagrange equations of the type of (12) in which all coefficients are real,‡ and in consequence the real and imaginary parts of the q's as typified by (24) must be separate and distinct solutions. When $\phi_k = 1$ and λ and ϕ_r $(r \neq k)$ are given by (23), this means that in the first solution the q's are typified by

$$\begin{aligned} q_r = \exp\left(-\frac{1}{4}\frac{\alpha_{kk}}{\mathrm{T}_k}t\right)\Big[(a_r)_k\cos\sqrt{\mu_k}\,t - \Big\{\frac{\alpha_{k1}}{\mu_k-\mu_1}\frac{(a_r)_1}{\mathrm{T}_1} + \\ +\frac{\alpha_{k2}}{\mu_k-\mu_2}\frac{(a_r)_2}{\mathrm{T}_2}+\ldots+0\times(a_r)_k+\ldots+ \\ +\frac{\alpha_{kN}}{\mu_k-\mu_N}\frac{(a_r)_N}{\mathrm{T}_N}\Big\}\frac{\sqrt{\mu_k}}{2}\sin\sqrt{\mu_k}\,t\Big], \end{aligned} \tag{25}$$

which is the real part of (24), while in the second solution they are typified by

$$\begin{aligned} q_r = \exp\left(-\frac{1}{4}\frac{\alpha_{kk}}{\mathrm{T}_k}t\right)\Big[(a_r)_k\sin\sqrt{\mu_k}\,t + \Big\{\frac{\alpha_{k1}}{\mu_k-\mu_1}\frac{(a_r)_1}{\mathrm{T}_1} + \\ +\frac{\alpha_{k2}}{\mu_k-\mu_2}\frac{(a_r)_2}{\mathrm{T}_2}+\ldots+0\times(a_r)_k+\ldots+ \\ +\frac{\alpha_{kN}}{\mu_k-\mu_N}\frac{(a_r)_N}{\mathrm{T}_N}\Big\}\frac{\sqrt{\mu_k}}{2}\cos\sqrt{\mu_k}\,t\Big], \end{aligned} \tag{26}$$

which is the imaginary part of (24). *The two solutions differ only in respect of phase*; for combining them in any proportion we obtain a

† The mode is indeterminate as regards absolute magnitude, so the postulate entails no restriction.

‡ Actually in relation to *small free* oscillations the terms $-\partial\mathfrak{T}/\partial q_k$ and Q_k in equation (12) are zero.

form for q_r which (apart from a multiplying constant which is irrelevant)† is identical with (25) except that t is replaced by $(t-t_0)$, t_0 being arbitrary.

To state this conclusion in another way—the a's to left of the N equations (16) may be replaced by q's if the ϕ's are defined as under:

$$\left.\begin{aligned} \phi_k &= \exp\left(-\frac{1}{4}\frac{\alpha_{kk}}{\mathbf{T}_k}t\right)\cos\sqrt{\mu_k}(t-t_0), \\ -\mathbf{T}_r\phi_r &= \exp\left(-\frac{1}{4}\frac{\alpha_{kk}}{\mathbf{T}_k}t\right)\sin\sqrt{\mu_k}(t-t_0)\times\frac{\alpha_{kr}\sqrt{\mu_k}}{2(\mu_k-\mu_r)} \quad (r\neq k). \end{aligned}\right\} \tag{27}$$

We observe that both of Lord Rayleigh's conclusions (§ 173) are confirmed; also that without great labour the method may be extended to obtain closer approximations (including terms of the order of $\mathbf{\Phi}^2$, $\mathbf{\Phi}^3$,..., etc.).

Example 2: Damped free oscillations of a simple system having three degrees of freedom

177. Formally this solution of the general problem of slightly damped vibrations is somewhat complicated: therefore it seems

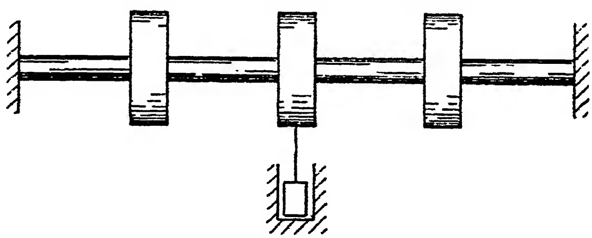

FIG. 34

desirable to consider first a system sufficiently simple to be treated exactly by orthodox methods.

A suitable example is presented in Fig. 34. A uniform shaft, clamped at each end, carries three thin disks each having moment of inertia I and can execute free vibrations under the influence (i) of the torsional stiffness of the shaft and (2) of small damping applied to the central disk as indicated. Den Hartog, in presenting this example (Ref. 1, § 30, Fig. 105), shows that the system is dynamically equivalent to a stretched string loaded with concentrated masses and

† Since the magnitude of a free oscillation is in all cases indeterminate except by the *initial conditions*.

executing transverse vibrations (Fig. 35), also to the 'spring-and-dashpot' system indicated in Fig. 36.†

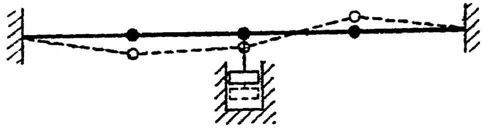

FIG. 35

178. For Fig. 34, if q_1, q_2, q_3 denote the angular rotations of the three disks and if an angular velocity $\dot{q}_2$ of the central disk entails a retarding torque of amount $k.\dot{q}_2$, the expressions for $\mathfrak{T}$, $\mathfrak{V}$, $\mathfrak{F}$ are

$$\left.\begin{aligned}\mathfrak{T} &= \tfrac{1}{2}I(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2),\\ \mathfrak{V} &= \frac{C}{l^2}\{q_1^2+q_2^2+q_3^2-q_2(q_3+q_1)\},\\ \mathfrak{F} &= \tfrac{1}{2}k\dot{q}_2^2,\end{aligned}\right\} \tag{28}$$

whence the Lagrange equations of type (12), simplified as before by suppression of the terms $-\dfrac{\partial\mathfrak{T}}{\partial q_k}$ and Q_k, are

$$\left.\begin{aligned}I\ddot{q}_1+\frac{C}{l^2}(2q_1-q_2) &= 0,\\ I\ddot{q}_2+k\dot{q}_2+\frac{C}{l^2}(-q_1+2q_2-q_3) &= 0,\\ I\ddot{q}_3+\frac{C}{l^2}(-q_2+2q_3) &= 0,\end{aligned}\right\} \tag{29}$$

—agreeing except as regards notation with Den Hartog's equations (89).‡ On the assumption stated in (13) and (14) they become

$$\left.\begin{aligned}\lambda^2a_1+B(2a_1-a_2) &= 0,\\ \lambda^2a_2+\lambda Aa_2+B(-a_1+2a_2-a_3) &= 0,\\ \lambda^2a_3+B(-a_2+2a_3) &= 0,\\ \text{where}\qquad A=\frac{k}{I},\qquad B &= \frac{C}{Il^2},\end{aligned}\right\} \tag{30}$$

and since on that understanding we have from (28)

$$\left.\begin{aligned}\mathbf{T} &= \tfrac{1}{2}I(a_1^2+a_2^2+a_3^2),\\ \mathbf{V} &= \frac{C}{l^2}\{a_1^2+a_2^2+a_3^2-a_2(a_3+a_1)\},\\ \mathbf{\Phi} &= \tfrac{1}{2}ka_2^2,\end{aligned}\right\} \tag{31}$$

† Fig. 35 is Den Hartog's Fig. 103, and Fig. 36 is Den Hartog's Fig. 104 (Ref. 1).
‡ I, k, C/l, q replace Den Hartog's m, c, T, x.

equations (30) can evidently be identified with (15) as relating to this particular example.

179. In an orthodox approach to the problem† we first eliminate a_1, a_2, a_3 from (30), thus obtaining the 'frequency equation'

$$\begin{vmatrix} \lambda^2+2B & -B & 0 \\ -B & \lambda^2+\lambda A+2B & -B \\ 0 & -B & \lambda^2+2B \end{vmatrix} = 0. \tag{32}$$

On expansion of the determinant this may be written as

$$(\lambda^2+2B)(\lambda^4+A\lambda^3+4\lambda^2B+ \\ +2\lambda AB+2B^2) = 0, \tag{33}$$

showing that in one mode of oscillation

$$\lambda^2 = -2B, \quad \text{simply.} \tag{34}$$

Substituting from (34) in (30), we deduce for the corresponding mode

$$a_1 : a_2 : a_3 = 1 : 0 : -1, \tag{35}$$

thus explaining why friction has no effect.

180. The other two solutions come from the second factor in (33), i.e. from the equation

$$\lambda^4+A\lambda^3+4\lambda^2B+2\lambda AB+2B^2 = 0, \tag{36}$$

in which all the coefficients are real. On this account the complex roots must occur in conjugate pairs; but even so calculation is difficult, and Den Hartog asserts that 'the classical method is unsuited to a practical solution of the problem'. Here we shall pursue it only on the simplifying assumption that *A is small.*

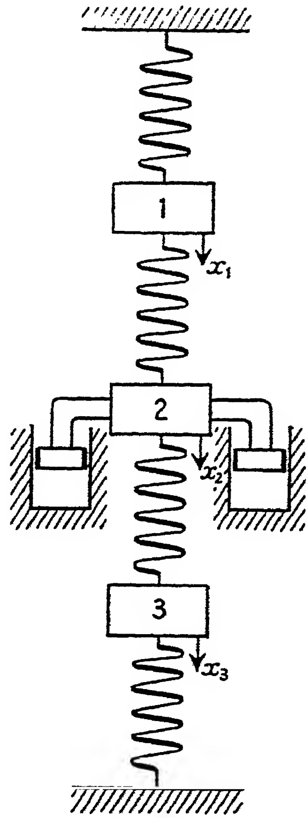

Fig. 36

When $A = 0$ the roots of (36) are

$$\lambda_1^2/B = -2+\sqrt{2}, \qquad \lambda_2^2/B = -2-\sqrt{2}. \tag{37}$$

Therefore in the general case we may assume that

$$\left.\begin{aligned} \lambda_1/\sqrt{B} &= \pm i\alpha+k_1A+\ldots(\text{higher powers of } A), \\ \lambda_2/\sqrt{B} &= \pm i\beta+k_2A+\ldots(\text{higher powers of } A), \\ \text{where} \qquad \alpha^2 &= 2-\sqrt{2}, \qquad \beta^2 = 2+\sqrt{2}, \end{aligned}\right\} \tag{38}$$

† Cf. Den Hartog, Ref. 1, § 30. A (= k/I) here replaces c/m; B (= T/Il^2) replaces T/ml; and λ replaces s.

and we can obtain an approximate solution by substituting in (36) with neglect of A^2 and higher powers of A. In this way we easily deduce that

$$k_1 = k_2 = -1.4\sqrt{B},$$

i.e. the other solutions of (36) are given approximately by

$$\left.\begin{aligned} \lambda_1 &= -\frac{A}{4} \pm i\alpha\sqrt{B}, \\ \lambda_2 &= -\frac{A}{4} \pm i\beta\sqrt{B}, \end{aligned}\right\} \tag{39}$$

α and β being defined by the third and fourth of (38).

Substituting in (30) with neglect as before of A^2 and higher powers of A, we deduce that in the mode corresponding with λ_1

$$a_1 : a_2 : a_3 = 1:(2+\lambda_1^2/B):1$$
$$= 1:\left[\sqrt{2} \mp i\frac{A}{2}\sqrt{\left(\frac{2-\sqrt{2}}{B}\right)}\right]:1,$$

and in the mode corresponding with λ_2 (40)

$$a_1 : a_2 : a_3 = 1:(2+\lambda_2^2/B):1$$
$$= 1:\left[-\sqrt{2} \mp i\frac{A}{2}\sqrt{\left(\frac{2+\sqrt{2}}{B}\right)}\right]:1.$$

Real forms can be deduced in the manner of § 176.

181. All of these results could have been obtained directly from the general formulae (23) of § 175, which presume a knowledge *for the system treated as frictionless* (*a*) of the natural frequencies and (*b*) of the corresponding modes. In this example, according to (37) and (34), the μ's in (19) are given by

$$\mu_1/B = 2-\sqrt{2}, \qquad \mu_2/B = 2+\sqrt{2}, \qquad \mu_3/B = 2, \tag{41}$$

and the corresponding normal modes can be deduced from (30) with the term in A suppressed. They are

$$\left.\begin{array}{llll} \phi_1 = 1: & a_1 = 1, & a_2 = \sqrt{2}, & a_3 = 1, \\ \phi_2 = 1: & a_1 = 1, & a_2 = -\sqrt{2}, & a_3 = 1, \\ \phi_3 = 1: & a_1 = 1, & a_2 = 0, & a_3 = -1, \end{array}\right\} \tag{42}$$

—in agreement (on the same understanding) with (40) and (35). When the ϕ's are thus defined, the forms assumed in this instance by (16) are

$$\left.\begin{aligned} a_1 &= \phi_1+\phi_2+\phi_3, \\ a_2 &= (\phi_1-\phi_2)\sqrt{2}, \\ a_3 &= \phi_1+\phi_2-\phi_3, \end{aligned}\right\} \tag{43}$$

whence the expressions (31) are equivalent to

$$\left.\begin{aligned} \mathbf{T} &= I(2\phi_1^2+2\phi_2^2+\phi_3^2), \\ \mathbf{V} &= 2\frac{C}{l^2}\{(2-\sqrt{2})\phi_1^2+(2+\sqrt{2})\phi_2^2+\phi_3^2\}, \\ \mathbf{\Phi} &= k(\phi_1-\phi_2)^2. \end{aligned}\right\} \tag{44}$$

In accordance with § 174 the introduction of the normal coordinates has reduced $\mathbf{T}$ and $\mathbf{V}$, *but not* $\mathbf{\Phi}$, to sums of squares. As defined in § 174, $\mathbf{T}_1 = 2I = \mathbf{T}_2$, $\mathbf{T}_3 = I$.

182. According to (44) the influence coefficients as defined in (20) and (22) of § 175 have values as under:

$$\alpha_{11} = \alpha_{22} = 2k = -\alpha_{12}, \qquad \alpha_{31} = \alpha_{32} = \alpha_{33} = 0.$$

Therefore according to (23), in the first mode as modified by damping

$$\lambda_1 = -\frac{k}{4I} \pm i\sqrt{(2-\sqrt{2})B}, \qquad \phi_1 = 1,$$

$$\phi_2 = \mp\frac{ik\sqrt{\mu_1}}{2I(\mu_1-\mu_2)} = \pm\frac{ik}{4I}\sqrt{\left(\frac{2-\sqrt{2}}{2B}\right)}, \qquad \phi_3 = 0,$$

and so according to (43), when terms involving k^2 are neglected, (45)

$$a_1 : a_2 : a_3 = 1 : \frac{\sqrt{2}(1-\phi_2)}{1+\phi_2} : 1$$

$$= 1 : \left(\sqrt{2} \mp \frac{ik}{2I}\sqrt{\frac{2-\sqrt{2}}{B}}\right) : 1.$$

In the second mode as thus modified

$$\lambda_2 = -\frac{k}{4I} \pm i\sqrt{(2+\sqrt{2})B}, \qquad \phi_2 = 1,$$

$$\phi_1 = \mp\frac{ik\sqrt{\mu_2}}{2I(\mu_2-\mu_1)} = \mp\frac{ik}{4I}\sqrt{\frac{2+\sqrt{2}}{2B}}, \qquad \phi_3 = 0,$$

therefore according to (43), when terms involving k^2 are neglected, (46)

$$a_1 : a_2 : a_3 = 1 : \frac{\sqrt{2}(\phi_1-1)}{\phi_1+1} : 1$$

$$= 1 : \left(-\sqrt{2} \mp \frac{ik}{2I}\sqrt{\frac{2+\sqrt{2}}{B}}\right) : 1.$$

In the third mode

$$\left.\begin{array}{l} \lambda_3 = \pm i\sqrt{(2B)}, \qquad \phi_3 = 1, \qquad \phi_1 = \phi_2 = 0, \\ \text{therefore according to (43)} \\ \qquad a_1 : a_2 : a_3 = 1 : 0 : -1. \end{array}\right\} \tag{47}$$

Since $A = k/I$, all of these results accord with those obtained by orthodox methods in §§ 179–80.

Damped torsional oscillations in rotating shafts

183. Having verified the formulae (23) as applied to this simple example, we now exemplify their use in relation to more complex systems, making use of numerical results which have been obtained in Chapter VIII.

It makes no difference to the torsional problem of that chapter (Fig. 33) whether we regard the clamped end as fixed or as rotating at some steady speed. When on the other hand damping devices are contemplated, their mode of operation has to be considered although, as will appear later, the computational problem is practically unaffected. If the clamped end of the shaft is fixed in space, the motion of any one mass can be damped by the agency of fluid friction operating in a fine clearance between the mass and a *fixed* surrounding sleeve; but when the unwanted oscillations are in relation to a 'datum' state of steady rotation (and this from a practical standpoint is the more important problem), then the sleeve too must rotate and its inertia should be sufficient to maintain its speed practically constant. This is the principle (e.g.) of the Lanchester vibration damper, which however operates by solid instead of fluid friction (Ref. 1, § 45). Here it concerns us only for the reason that we shall have to attach numerical values to the damping terms in the Lagrange equations.

184. Fig. 37 exhibits (diagrammatically) one of the inertial disks of Fig. 33 surrounded by a sleeve S as described above. When the clamped end of the shaft does not rotate, this sleeve also will be fixed, and we may describe its operation as consisting in the imposition of a retarding couple $k\dot{\theta}$ on the motion (θ) of the disk, k being positive and known. Suppose that *the same* sleeve S is fitted to the same system now rotating with uniform angular velocity ω: then in the absence of vibration we may assert that the sleeve too will rotate with steady speed ω (since fluid friction will oppose any relative motion of shaft and sleeve). If now in a torsional oscillation the

disk in Fig. 37 has an angular velocity $(\omega+\dot{\theta})$ and the sleeve an angular velocity $(\omega+\dot{\phi})$, the angular velocity of the disk relative to the sleeve will be $(\dot{\theta}-\dot{\phi})$ and there will be, on the above understanding, a retarding couple $k(\dot{\theta}-\dot{\phi})$ on the motion (θ) of the disk, therefore an accelerating couple $k(\dot{\theta}-\dot{\phi})$ on the motion of the sleeve. The latter will be governed by the equation

$$S\ddot{\phi} = k(\dot{\theta}-\dot{\phi}), \tag{48}$$

S denoting the moment of inertia of the sleeve.

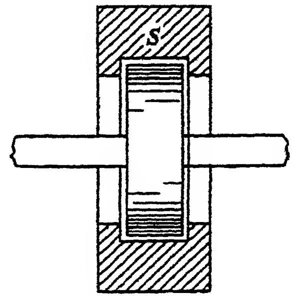

FIG. 37

185. Now θ, in all cases of importance, will be oscillatory with small damping factor: therefore in this general treatment no serious inaccuracy will be entailed if we assume it to vary in accordance with a purely harmonic time-factor e^{ipt}, so that

$$\dot{\theta} = ip\theta, \qquad \ddot{\theta} = -p^2\theta. \tag{i}$$

Then according to (48)
$$\dot{\phi} = \frac{k\dot{\theta}}{k+ipS}, \tag{ii}$$

if the initial conditions (i.e. the complementary function) are disregarded. Consequently the (negative) couple tending to increase $\dot{\theta}$ is

$$-k(\dot{\theta}-\dot{\phi}) = -k\dot{\theta}\times\frac{ipS}{k+ipS} \quad \text{by (ii)}$$

$$= k\theta\times\frac{p^2S}{k+ipS} \quad \text{by (i)}$$

$$= \theta\times\frac{k^2p^2S-ikp^3S^2}{p^2S^2+k^2}. \tag{49}$$

The real term in (49), viz.

$$\theta\times k^2p^2S/(p^2S^2+k^2),$$

is the accelerating torque which would neutralize an increase

$$\Delta I = k^2S/(p^2S^2+k^2) \tag{50}$$

in the inertia of the disk. The imaginary term, viz.

$$\theta\times -ikp^3S^2/(p^2S^2+k^2) = -\dot{\theta}\times kp^2S^2/(p^2S^2+k^2), \quad \text{by (i)},$$

represents (cf. § 184) damping equivalent to that of a fixed sleeve operating on a shaft fixed at one end, but with k replaced by

$$k' = k/(1+k^2/p^2S^2). \tag{51}$$

186. Since k, in all cases of practical importance, will be small in relation to pS, with close approximation we may neglect both the 'effective inertia' ΔI and the difference according to (51) between k and k'. In any event this analysis shows that we can transform the problem for a rotating shaft into an equivalent 'static' problem of the kind discussed in §§ 174–6.

It is thus unnecessary to particularize further the two vibration problems of § 183. Whether the shaft be fixed at one end or rotating, we can assume that 'effective' values have been specified for each inertia (I) and for its associated damping constant (k).

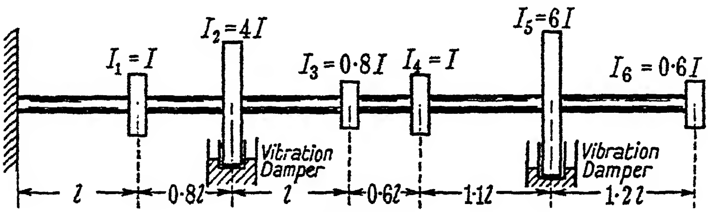

FIG. 38

Example 3: The torsional system of Fig. 33 as modified by the operation of vibration dampers

187. Fig. 38 exhibits (diagrammatically) the torsional system of Fig. 33, Chapter VII, modified by the provision of 'vibration dampers' operating on the disks denoted by I_2 and I_5. The intensity of the damping is assumed to be such that motion of I_2 is influenced by a retarding torque $k\dot{q}_2$ and motion of I_5 by a retarding torque $2k\dot{q}_5$, q_2 and q_5 being the angular displacements of I_2 and I_5. I.e. the 'dissipation function' $\mathfrak{F}$ (cf. § 172 and footnote) is given by

$$2\mathfrak{F} = k(\dot{q}_2^2+2\dot{q}_5^2). \tag{52}$$

As before our treatment presumes that k is small.

188. Assuming as in § 172 that every coordinate varies in accordance with a time-factor $e^{\lambda t}$, so that

$$q_k = a_k e^{\lambda t} \quad (a_k \text{ constant}), \qquad (13)\,bis$$

we have from (52) $\quad \Phi = \frac{1}{2}k(\dot{q}_2^2+2\dot{q}_5^2)$,

or

$$\Phi = \tfrac{1}{2}k\alpha^2(a_2^2+2a_5^2) \tag{53}$$

in the notation of § 148 (Chap. VII). The forms of $\mathbf{V}$ and $\mathbf{T}$ are unaffected by friction, so can be taken from § 148 with the single

modification that we do not now assume a_1 to have the value unity. In the typical Lagrange equation of free oscillation, viz.

$$\lambda^2 T_k + \lambda\Phi_k + V_k = 0, \qquad (15)\,bis$$

we may suppress 'dimensional' terms and write

$$\left.\begin{aligned} 2\mathbf{T} &= a_1^2+4a_2^2+0{\cdot}8a_3^2+a_4^2+6a_5^2+0{\cdot}8a_6^2, \\ 2\mathbf{V} &= \mathbf{A}[2{\cdot}25(a_1^2+a_2^2)+2{\cdot}\dot{6}a_3^2+2{\cdot}\dot{5}\dot{7}a_4^2+1{\cdot}7\dot{4}\dot{2}a_5^2+0{\cdot}8\dot{3}a_6^2- \\ &\quad -2(1{\cdot}25a_1a_2+a_2a_3+1{\cdot}\dot{6}a_3a_4+0{\cdot}\dot{9}\dot{0}a_4a_5+0{\cdot}8\dot{3}a_5a_6)], \end{aligned}\right\} \qquad (54)$$

provided that at the same time we substitute for (53) the expression

$$2\mathbf{\Phi} = B(a_2^2+2a_5^2),$$

where **A** and **B** are dimensional factors defined by

$$\mathbf{A} = C/lI_1, \qquad \mathbf{B} = k/I_1, \qquad (55)$$

in which C, l and I_1 have the same significance as in § 148.

By this procedure we simplify the numerical work and at the same time retain the greatest possible measure of generality in our solution. A similar procedure was employed in § 148, where $p^2I_1\,l/C$ was represented by a single symbol λ: accordingly values of λ as determined in Chapter VIII are values of $\mu/\mathbf{A}$ as defined in (17) and (55), and we have

$$\mu_k = \mathbf{A}\lambda_k, \qquad (56)$$

where λ_k typifies the 'characteristic values' of Chapters VII and VIII.

189. The μ's for this problem are thus known already from the results of Chapter VIII, and the corresponding modes have also been determined, so we can attach known values to the coefficients of type $(a_k)_1$ in (16). Table XXXVI collects the necessary data: its second column (for example) records the results of line 20 of Table XXXI, Chapter VIII, and from its first line (for example) we can deduce the expression

$$\begin{aligned} a_1 = \phi_1+1{\cdot}004014_4\phi_2+0{\cdot}258318_8\phi_3+0{\cdot}262351_7\phi_4+ \\ +1{\cdot}346877_4\phi_5+0{\cdot}0579700\phi_6, \end{aligned} \qquad (57)$$

which is of the type of (16). At the top of each column is recorded the corresponding quantity $\mu/\mathbf{A}$ $(=\lambda)$ as given by the 'Rayleigh limit' in Chapter VIII, and at the bottom of every column the corresponding quantity of type $2\mathbf{T}_r$ (§ 174), which is the quantity denoted by $\sum_T/10^3$ in Table XXIX and by $\sum_T/10^6$ in Table XXXI of Chapter VIII.

TABLE XXXVI. *Summary of Results obtained in Chapter VIII*

	First mode $\mu_1/A = 0{\cdot}025861$	*Second mode* $\mu_2/A = 0{\cdot}223969$	*Third mode* $\mu_3/A = 1{\cdot}1124246$	*Fourth mode* $\mu_4/A = 1{\cdot}320642$	*Fifth mode* $\mu_5/A = 2{\cdot}457992$	*Sixth mode* $\mu_6/A = 4{\cdot}912809$
a_1	1	$1{\cdot}004014_4$	$0{\cdot}258318_8$	$0{\cdot}262351_7$	$1{\cdot}346877_4$	0·0579700
a_2	1·77927	$1{\cdot}627318_6$	$0{\cdot}235146_6$	$0{\cdot}195103_3$	$-0{\cdot}221121_5$	−0·1236400
a_3	2·56899	$0{\cdot}919130_1$	$-0{\cdot}839693_8$	$-0{\cdot}919200_1$	$+0{\cdot}015653_9$	2·0789600
a_4	3·01088	$0{\cdot}439762_6$	$-1{\cdot}036192_2$	$-1{\cdot}005067_1$	$+0{\cdot}140984_4$	−1·5019800
a_5	3·73538	$-0{\cdot}602679_6$	$-0{\cdot}128461_4$	$0{\cdot}297613_6$	$-0{\cdot}010340_8$	+0·0496400
a_6	3·83048	$-0{\cdot}767874_8$	$1{\cdot}890370_3$	$-1{\cdot}111429_4$	$+0{\cdot}007615_3$	−0·0133600
	$2T_1 = 123{\cdot}464817$	$2T_2 = 15{\cdot}1658193$	$2T_3 = 4{\cdot}8834810$	$2T_4 = 3{\cdot}4268559$	$2T_5 = 2{\cdot}0357665$	$2T_6 = 5{\cdot}7930175$

TABLE XXXVII. *Values of* α_{kr}/B

$r =$ \ $k =$	1	2	3	4	5	6
1	31·07193	−1·607035	−0·541315	2·570541	−0·476032	0·150860
2	−1·607035	3·37461	0·537501	−0·0412361	−0·353358	−0·261036
3	−0·541315	0·537501	0·0882985	−0·0305858	−0·0500453	−0·0418272
4	2·570541	−0·0412361	−0·0305858	0·215213	−0·0498826	0·0054245
5	−0·476032	−0·352258	−0·0500453	−0·0498826	0·0504457	0·0266841
6	0·150860	−0·261036	−0·0418272	0·0054245	0·0266841	0·0202151

190. Substituting for a_2 and a_5 their expressions of type (57) as derived from Table XXXVI, we transform the expression (55) for $\mathbf{\Phi}$ into an expression involving ϕ's in place of a's; and then, defining α_{kr} by

$$\alpha_{kr} \equiv \frac{\partial^2 \mathbf{\Phi}}{\partial \phi_r \, \partial \phi_k} \tag{(22) bis}$$

as in § 175, we can deduce Table XXXVII.

The work now becomes a matter of straightforward substitution in the formulae (23), followed by calculation of the a's from the ϕ's in accordance with Table XXXVI. It is naturally lengthy, but presents no serious difficulty. We shall limit our description to the computations which start from the first normal mode (ϕ_1).

Modified first mode and frequency

191. From column 1 of Table XXXVI we have

$$\mu_1/\mathbf{A} = 0{\cdot}025861, \qquad 2\mathbf{T}_1 = 123{\cdot}464817,$$

and from Table XXXVII

$$\alpha_{11} = 31{\cdot}07193\mathbf{B}.$$

Therefore according to the first of (23)

$$\lambda_1 = -\frac{31{\cdot}07193\mathbf{B}}{2\times 123{\cdot}464817} \pm i\sqrt{(0{\cdot}025861\mathbf{A})}$$

$$= -0{\cdot}12583\mathbf{B} \pm i \times 0{\cdot}160812\sqrt{\mathbf{A}}. \tag{58}$$

We postulate that $\phi_1 = 1$ in the first mode as modified by damping, and then from the second of (23) we have

$$2\mathbf{T}_2(\mu_1 - \mu_2)\phi_2 = \pm i\alpha_{12}\sqrt{\mu_1},$$

in which values of $\mathbf{T}_2, \mu_1, \mu_2, \alpha_{12}$ must be taken from Tables XXXVI and XXXVII. The result is

$$\left.\begin{aligned}
\phi_1 &= 1,\\
\phi_2 &= \pm i\frac{(-1{\cdot}607035\mathbf{B})\times(0{\cdot}160812\sqrt{\mathbf{A}})}{15{\cdot}1658193\times(-0{\cdot}198108\mathbf{A})}\\
&= \pm i\times 0{\cdot}086014\mathbf{B}/\sqrt{\mathbf{A}},\\
&\text{and we have by the same procedure}\\
\phi_3 &= \pm i\times 0{\cdot}016406\mathbf{B}/\sqrt{\mathbf{A}},\\
\phi_4 &= \mp i\times 0{\cdot}093165\mathbf{B}/\sqrt{\mathbf{A}},\\
\phi_5 &= \pm i\times 0{\cdot}015461\mathbf{B}/\sqrt{\mathbf{A}},\\
\phi_6 &= \mp i\times 0{\cdot}00085694\mathbf{B}/\sqrt{\mathbf{A}}.
\end{aligned}\right\} \tag{59}$$

Finally, substituting these values in (57) and in the other similar expressions which are derivable from Table XXXVI, we deduce that

$$\left.\begin{aligned} a_1 &= 1 \pm i \times 0\cdot08692\mathbf{B}/\sqrt{\mathbf{A}}, \\ a_2 &= 1\cdot77927 \pm i \times 0\cdot12229\mathbf{B}/\sqrt{\mathbf{A}}, \\ a_3 &= 2\cdot56899 \pm i \times 0\cdot15196\mathbf{B}/\sqrt{\mathbf{A}}, \\ a_4 &= 3\cdot01088 \pm i \times 0\cdot11793\mathbf{B}/\sqrt{\mathbf{A}}, \\ a_5 &= 3\cdot73538 \mp i \times 0\cdot08188\mathbf{B}/\sqrt{\mathbf{A}}, \\ a_6 &= 3\cdot83048 \pm i \times 0\cdot06864\mathbf{B}/\sqrt{\mathbf{A}}. \end{aligned}\right\} \tag{60}$$

192. These computations, with similar determinations of the other modes and frequencies as modified by damping, have been made by Miss A. Pellew. Her complete solution for the system shown in Fig. 38 (*on the assumption that k and therefore* **B** *are small*) is presented in Table XXXVIII.

The solution can be checked by substitution in the Lagrange equations of motion as modified to take account of damping. Corresponding with (28) of Chapter VII we have according to (15), in the notation of this chapter,

$$\left.\begin{aligned} \lambda^2 a_1 + \mathbf{A}[2\cdot25a_1 - 1\cdot25a_2] &= 0, \\ 4\lambda^2 a_2 + \mathbf{B}\lambda a_2 + \mathbf{A}[2\cdot25a_2 - (1\cdot25a_1 + a_3)] &= 0, \\ 0\cdot8\lambda^2 a_3 + \mathbf{A}[2\cdot\dot{6}a_3 - (a_2 + 1\cdot\dot{6}a_4)] &= 0, \\ \lambda^2 a_4 + \mathbf{A}[2\cdot\dot{5}\dot{7}a_4 - (1\cdot\dot{6}a_3 + 0\cdot\dot{9}\dot{0}a_5)] &= 0, \\ 6\lambda^2 a_5 + 2\mathbf{B}\lambda a_5 + \mathbf{A}[1\cdot7\dot{4}\dot{2}a_5 - (0\cdot\dot{9}\dot{0}a_4 + 0\cdot8\dot{3}a_6)] &= 0, \\ 0\cdot8\lambda^2 a_6 + 0\cdot8\dot{3}\mathbf{A}(a_6 - a_5) &= 0, \end{aligned}\right\} \tag{61}$$

and in these equations, if values for λ and for the a's are taken from any of the six solutions in Table XXXVIII, both real and imaginary terms cancel very closely when $\mathbf{B}^2$ and $\mathbf{B}^3$ are treated as negligible.

RECAPITULATION

193. This chapter leads from work already published to fields in which investigation is still proceeding. It exploits the solution found in Chapter VIII, showing how account can be taken of frictional forces ('damping'), also of external forces whether steady or pulsating, once full knowledge is available of the natural frequencies and of the mode associated with each. Its greater part describes work done in collaboration with Miss A. Pellew to exemplify methods which were indicated in a recent paper (Ref. 3). The matter is not such as can be briefly summarized.

TABLE XXXVIII. *Complete Solution for Small Damping of the Torsional Problem of Fig.* 38

$\lambda_1 = (-0{\cdot}12583\,\mathbf{B} \pm i\,0{\cdot}16081\sqrt{\mathbf{A}})$	$\lambda_2 = (-0{\cdot}11126\,\mathbf{B} \pm i\,0{\cdot}47325\sqrt{\mathbf{A}})$	$\lambda_3 = (-0{\cdot}0090405\,\mathbf{B} \pm i\,1{\cdot}05471\sqrt{\mathbf{A}})$
$a_1 = 1 \pm i\,0{\cdot}08692\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_1 = 1{\cdot}0040 \pm i\,0{\cdot}00476\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_1 = 0{\cdot}25832 \pm i\,0{\cdot}07592\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_2 = 1{\cdot}77927 \pm i\,0{\cdot}12229\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_2 = 1{\cdot}62732 \mp i\,0{\cdot}07687\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_2 = 0{\cdot}23515 \pm i\,0{\cdot}06515\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_3 = 2{\cdot}56899 \pm i\,0{\cdot}15196\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_3 = 0{\cdot}94913 \mp i\,0{\cdot}02539\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_3 = -0{\cdot}83969 \mp i\,0{\cdot}00809\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_4 = 3{\cdot}01088 \pm i\,0{\cdot}11793\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_4 = 0{\cdot}43976 \mp i\,0{\cdot}03975\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_4 = -1{\cdot}03619 \mp i\,0{\cdot}04004\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_5 = 3{\cdot}73538 \mp i\,0{\cdot}08188\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_5 = -0{\cdot}60268 \mp i\,0{\cdot}10722\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_5 = -0{\cdot}12846 \mp i\,0{\cdot}02790\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_6 = 3{\cdot}83048 \pm i\,0{\cdot}06864\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_6 = -0{\cdot}76787 \mp i\,0{\cdot}23549\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_6 = 1{\cdot}89037 \mp i\,0{\cdot}09874\,\mathbf{B}/\sqrt{\mathbf{A}}$
$\lambda_4 = (-0{\cdot}031401\,\mathbf{B} \pm i\,1{\cdot}14919\sqrt{\mathbf{A}})$	$\lambda_5 = (-0{\cdot}01239\,\mathbf{B} \pm i\,1{\cdot}56780\sqrt{\mathbf{A}})$	$\lambda_6 = (-0{\cdot}0017448\,\mathbf{B} \pm i\,2{\cdot}21649\sqrt{\mathbf{A}})$
$a_1 = 0{\cdot}26235 \pm i\,0{\cdot}04002\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_1 = 1{\cdot}34688 \mp i\,0{\cdot}02737\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_1 = 0{\cdot}05797 \pm i\,0{\cdot}00729\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_2 = 0{\cdot}19510 \pm i\,0{\cdot}01460\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_2 = -0{\cdot}22412 \mp i\,0{\cdot}03730\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_2 = -0{\cdot}12364 \mp i\,0{\cdot}01589\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_3 = -0{\cdot}91920 \pm i\,0{\cdot}07356\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_3 = 0{\cdot}01565 \pm i\,0{\cdot}00050\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_3 = 2{\cdot}07896 \mp i\,0{\cdot}00282\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_4 = -1{\cdot}00507 \pm i\,0{\cdot}09414\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_4 = 0{\cdot}14098 \pm i\,0{\cdot}02231\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_4 = -1{\cdot}50198 \pm i\,0{\cdot}00395\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_5 = 0{\cdot}29761 \pm i\,0{\cdot}07491\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_5 = -0{\cdot}01034 \mp i\,0{\cdot}00404\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_5 = +0{\cdot}04964 \pm i\,0{\cdot}00778\,\mathbf{B}/\sqrt{\mathbf{A}}$
$a_6 = -1{\cdot}11143 \pm i\,0{\cdot}00782\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_6 = 0{\cdot}00761 \pm i\,0{\cdot}00276\,\mathbf{B}/\sqrt{\mathbf{A}}$	$a_6 = -0{\cdot}01336 \mp i\,0{\cdot}00207\,\mathbf{B}/\sqrt{\mathbf{A}}$

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X

CONTINUOUS SYSTEMS. I. PROBLEMS OF EQUILIBRIUM

194. Whether the problem be concerned with equilibrium or with vibrations, the size of the Relaxation Table is conditioned by the number (N) of the independent variables or coordinates, i.e. by the 'freedom' of the system. Every coordinate calls for a special column in the table, so the total number of columns will in no case be less than N.† If N exceeds 15 or 20 the labour of relaxation may be very great: if it were really large, the method would be quite impracticable.

Now in many of the problems which normally confront the engineer, so far from being small of the order of 15 or 20 the freedom of the system is in fact *infinite*. This is true whenever *continuous* elastic bodies are involved,—e.g. girders, shafts, plates and shells: indeed, it is only in virtue of artificial simplifications made in the theory that frameworks (which are assemblages of continuous members) can be treated as systems of finite freedom. By other simplifications (justified in the theory of elasticity) girders are in effect reduced to a single line of particles—the 'central line' of flexural theory; but even so they have infinite freedom, since a bent form is not specified completely unless a value is attached to the deflexion at every section. A problem really two- or three-dimensional may by separation of the independent variables be reducible to simpler form: it will not be brought within the scope of relaxation methods unless the order of freedom can be rendered *finite and fairly small*.

The purpose of this chapter is to examine ways in which such simplification can be effected. We are accustomed in experiment to 'determine' curves of deflexion by measurements made at ten or twelve different sections of a girder, so it is clear that determination can have in practical work a meaning very different from its mathematical connotation. A curve, in fact, may be *sufficiently* determined by a finite number of ordinates, and this assumption is the basis of the branch of mathematics known as the Calculus of Observations. In this chapter we seek to extend the scope of relaxation methods by utilizing the results of that calculus.

† It may be greater than N; e.g. in Table V, Chap. II, where some columns relate to actions which do not call for liquidation but of which it is desirable to keep account.

Formulae for approximate integration

195. Although in fact the value of a definite integral, typified by

$$\int_0^l y\,dx, \tag{1}$$

depends upon the value of the integrand (y) at every point in the range ($0 \leqslant x \leqslant l$), nevertheless an approximation to its correct value can (provided that the function y is reasonably 'smooth') be formulated in terms of $y_0, y_1, y_2,..., y_n$, the values assumed by y at points which subdivide the range into n equal parts. 'Simpson's rule' is a familiar example of such formulae for approximate integration: in effect it assumes that over any two contiguous parts of the range the integrand y can be replaced by a quadratic function of x, so in fact it is the formula appropriate to the case where $n = 2$. But corresponding formulae can be constructed in which n has higher values, and as might be expected their accuracy (in general) increases with n. A very useful summary of these integration formulae has been given by W. G. Bickley (Ref. 2).

Using the formula appropriate to any value n, in effect (as will appear later, e.g. in § 205) we reduce the order of the freedom of our system to

$$N = n+1-k, \tag{2}$$

where k is the number of the imposed terminal conditions: therefore from the standpoint of approximate computation we must compromise between the conflicting requirements of accuracy and of speed. In this chapter we shall employ the formulae appropriate to $n = 6$, which for many purposes will be a satisfactory compromise; but it should be emphasized that similar treatment is possible using other values of n.†

196. Corresponding with $n = 6$ we have the approximate formula of integration

$$\frac{1}{l}\int_0^l y\,dx \approx \frac{1}{840}\{41(y_0+y_6)+216(y_1+y_5)+27(y_2+y_4)+272y_3\} \tag{3}$$

(Ref. 2, formula I.6). Suppose now, for example, that we want an approximation to the exact value of

$$\int_0^1 x\sin^2\pi x\,dx \quad (y = x\sin^2\pi x),$$

† For the material of §§ 196–200 and Table XXXIX I am indebted to Dr. Bickley, who has very kindly communicated to me some results not yet published.

which is easily shown to be 0·25. Then we have

$x =$	0	1/6	2/6	3/6	4/6	5/6	6/6
$\sin \pi x =$	0	1/2	$\sqrt{3}/2$	1	$\sqrt{3}/2$	1/2	0
$\sin^2 \pi x =$	0	0·25	0·75	1·0	0·75	0·25	0
$x \sin^2 \pi x =$	0	1/24	6/24	12/24	12/24	5/24	0

and accordingly the approximation given by (3) is

$$\int^1 x \sin^2 \pi x \, dx$$

$$\approx \frac{41 \times 0 + 216 \times (1+5) + 27 \times (6+12) + 272 \times 12}{140 \times 6 \times 24} = 0{\cdot}250297. \quad (4)$$

Thus the error entailed by (3) in this instance is about 0·12 per cent.

Formulae for approximate differentiation

197. The formula (3) is all that we require for evaluating (approximately) $\mathfrak{V}$ as expressed in (12) of this chapter. But the expression for $\mathfrak{U}$ in (11), and the expression for $\mathfrak{V}_2$ in (6) of § 229, involve the differential coefficients d^2y/dx^2 and dy/dx, whereas here we require to express them (approximately) in terms of y_0, $y_1, \ldots, y_N$. Therefore before (3) can be utilized we must have in addition formulae for approximate differentiation.

Table XXXIX presents the requisite formulae in tabular form,—again in relation to the value $n = 6$. For brevity, let $y_r^{(m)}$ stand for the value of $(d^m y/dx^m)_r$, i.e. for the mth differential of y at the point $x/l = r/n$. Then, h denoting the quantity l/n, we want (for $n = 6$) values of A_0, $A_1, \ldots, A_6$ in the expression

$$\frac{h^m}{m!} y_r^{(m)} \approx \frac{1}{6!}(A_0 y_0 + A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_4 + A_5 y_5 + A_6 y_6), \quad (5)$$

and these values, for $m = 1, 2, \ldots, 6$ and for $r = 0, 1, 2, \ldots, 6$, are recorded in the table with an indication of the order (E) of the error involved in each case by the use of (5).

198. Table XXXIX, as having general application, is placed at the end of this book. Here we are concerned only with the sections relating to $m = 1$ and to $m = 2$, and these (except as regards the column which relates to the error E) are reproduced below.

When $m = 1$ the formula (5) may be written as

$$6!\,h\left(\frac{dy}{dx}\right)_r \approx A_0y_0+A_1y_1+A_2y_2+A_3y_3+A_4y_4+A_5y_5+A_6y_6, \tag{6}$$

where, for $r = 0, 1, 2,..., 6$, the A's are given in Table XL.

TABLE XL

$r =$	A_0	A_1	A_2	A_3	A_4	A_5	A_6
0	−1,764	4,320	−5,400	4,800	−2,700	864	−120
1	−120	−924	1,800	−1,200	600	−180	24
2	24	−288	−420	960	−360	96	−12
3	−12	108	−540	0	540	−108	12
4	12	−96	360	−960	420	288	−24
5	−24	180	−600	1,200	−1,800	924	120
6	120	−864	2,700	−4,800	5,400	−4,320	1,764

When $m = 2$, the formula (5) may be written as

$$\frac{6!}{2}h^2\left(\frac{d^2y}{dx^2}\right)_r \approx A_0y_0+A_1y_1+A_2y_2+A_3y_3+A_4y_4+A_5y_5+A_6y_6, \tag{7}$$

where, for $r = 0, 1, 2,..., 6$, the A's are given in Table XLI.

TABLE XLI

$r =$	A_0	A_1	A_2	A_3	A_4	A_5	A_6
0	1,624	−6,264	10,530	−10,160	5,940	−1,944	274
1	274	−294	−510	940	−570	186	−26
2	−26	456	−840	400	30	−24	4
3	4	−54	540	−980	540	−54	4
4	4	−24	30	400	−840	456	−26
5	−26	186	−570	940	−510	−294	274
6	274	−1,944	5,940	−10,160	10,530	−6,264	1,624

199. To illustrate the use of (6) and (7) we shall calculate approximate values of the definite integrals

$$\int^1 xy'^2\,dx \quad \text{and} \quad \int^1 xy''\cdot dx$$

in terms of $y_0, y_1,..., y_6$, for the case in which $y = \sin \pi x$. (It is easy to show that the exact values are $\pi^2/4$ and $\pi^4/4$ respectively.) As a first step we proceed to calculate y' ($= dy/dx$) by means of (6) *without recourse to the orthodox process of differentiation* (which of course can be applied without difficulty to $\sin \pi x$).

For the calculation of (dy/dx) at the end $r = 0$ we have

$x =$	0	1/6	2/6	3/6	4/6	5/6	6/6
$y = \sin \pi x =$	0	1/2	$\sqrt{3}/2$	1	$\sqrt{3}/2$	1/2	0
A from § 198 $=$	-1764	4320	-5400	4800	-2700	864	-120
$Ay =$	0	2160	$-2700\sqrt{3}$	4800	$-1350\sqrt{3}$	432	0

Therefore

$$\sum (Ay) = 377{\cdot}1972... = 5!\left(\frac{dy}{dx}\right)_0 \quad \text{according to (6),}$$

since h here $= 1/6$. So our approximate formula gives the estimate $(dy/dx)_0 \approx 3{\cdot}14331$, which agrees well with the correct value (π).

Proceeding in the same way for other values of r, we obtain the second line of the table which follows. The third line is self-explanatory.

$x =$	0	1/6	2/6	3/6	4/6	5/6	6/6
$y' = \dfrac{dy}{dx} =$	3·14331	2·72050	1·57084	0	$-1{\cdot}57084$	$-2{\cdot}72050$	$-3{\cdot}14331$
$xy'^2 =$	0	1·23352	0·82251	0	1·64503	6·16760	9·88040

Finally, using (3) in the manner of § 196, we obtain the approximation

$$\int_0^1 xy'^2\,dx \quad (y = \sin \pi x) \quad \approx \frac{2070{\cdot}36190}{840} = 2{\cdot}46472.... \tag{8}$$

The correct value $\pi^2/4 = 2{\cdot}46740...$, so the error entailed by (3) and (6) in this instance is $-0{\cdot}11$ per cent.

200. From (7), by a similar process, we arrive at the following table:

$x =$	0	1/6	2/6	3/6	4/6	5/6	6/6
$y'' = \dfrac{d^2y}{dx^2} =$	$-0{\cdot}05682$	$-4{\cdot}93070$	$-8{\cdot}54803$	$-9{\cdot}86930$	$-8{\cdot}54803$	$-4{\cdot}93070$	$-0{\cdot}05682$
$xy''^2 =$	0	4·05197	24·35627	48·70154	48·71254	20·25984	0·00323

Then we have from (3)

$$\int_0^1 xy''^2\,dx \quad (y = \sin \pi x) \quad \approx \frac{20471{\cdot}16...}{840} = 24{\cdot}3704.... \tag{9}$$

The correct value $\pi^4/4 = 24{\cdot}3523...$, so the error entailed by (3) and (7) in this instance is 0·074 per cent.

Use of the approximate formulae in problems relating to the equilibrium of straight girders

201. In these examples, y being specified, we could attach numerical values to $y_0, y_1,..., y_6$. But we can also use the formulae (3) and (5) to express any definite integral of the type

$$\int_0^l f(x)\frac{d^m y}{dx^m}\,dx, \tag{10}$$

$f(x)$ being specified, in terms of $y_0, y_1,..., y_6$ *regarded as unknowns*. The values of $f(x)$ at the points of subdivision (say, $f_0, f_1,..., f_6$) will enter into the general formulae, which accordingly are lengthy; but in particular examples ($f_0, f_1,..., f_6$ being specified) the calculations are quite simple, as will now appear.

202. In the approximate theory of flexure (*Elasticity*, chap. VI), if at any section x of a straight girder y denotes the transverse deflexion, w the line-intensity of the transverse loading, and B the (specified) flexural rigidity, then the elastic strain-energy can be represented by

$$\mathfrak{U} = \frac{1}{2}\int_0^L B\left(\frac{d^2y}{dx^2}\right)^2 dx \tag{11}$$

and the potential energy of the applied loading by

$$\mathfrak{V} = -\int_0^L wy\,dx, \tag{12}$$

when the ends are at $x = 0$ and at $x = L$. We know (§ 97) that in the required configuration of equilibrium the total potential energy $(\mathfrak{U}+\mathfrak{V})$ assumes its minimum value; so the problem of an exact solution is to find that form for y which makes the quantity

$$\mathfrak{U}+\mathfrak{V} = \frac{1}{2}\int_0^L B\left(\frac{d^2y}{dx^2}\right)^2 dx - \int_0^L wy\,dx$$

as small as possible.

Using orthodox methods (the Calculus of Variations), we have to make

$$\delta(\mathfrak{U}+\mathfrak{V}) = \int_0^L \left(B\frac{d^2y}{dx^2}\frac{d^2\,\delta y}{dx^2} - w\,\delta y\right) dx = 0 \tag{13}$$

for all permissible variations δy, and this requirement leads in virtue of the terminal conditions to the relation

$$\frac{d^2}{dx^2}\left(B\frac{d^2y}{dx^2}\right) = w, \tag{14}$$

—a familiar equation of which, in general, it is not easy to obtain an exact solution.

We can, however, bring the problem within the scope of relaxation methods, and so arrive at a *sufficiently* exact solution, if we replace the exact expressions (11) and (12) by approximate expressions, formulated in accordance with §§ 196–200, which involve only y_0, y_1,..., y_6. Then, in place of (13), we shall have a finite number of equations, typified by

$$\frac{\partial}{\partial y_r}(\mathfrak{U}+\mathfrak{V}) = 0, \tag{15}$$

which are the conditions for a minimum value of $\mathfrak{U}+\mathfrak{V}$ *regarded as a function of* y_0, y_1,..., y_6. The terminal conditions will of course have to be taken into account: this part of the problem will be best explained by means of a particular example.

A numerical example: Deflexions and bending moments in a clamped beam

203. Consider a straight beam clamped at either end (so that

$$y = \frac{dy}{dx} = 0 \quad \text{both when } x = 0 \text{ and when } x = L) \tag{16}$$

and having a flexural rigidity defined by

$$B = B_0(1-\tfrac{1}{2}z^2) = B_0\beta(z), \quad \text{say}, \tag{17}$$

where z stands for x/L and B_0 is the value of B at the end $(z = 0)$. Let it be subjected to a transverse loading w defined by

$$w = w_0(1-2z^2) = w_0 W(z), \quad \text{say}, \tag{18}$$

so that w_0 measures the intensity of loading at the end $(z = 0)$; and let the consequent deflexion y be expressed in the form

$$y = \frac{w_0 L^4}{B_0} Y(z), \tag{19}$$

so that $Y(z)$—the quantity which has to be determined—is purely numerical.

Then according to (11) we have

$$\mathfrak{U} = \frac{1}{2}\frac{w_0^2 L^5}{B_0}\int_0^1 \beta(z)\{Y''(z)\}^2\,dz,$$

and according to (12)

$$\mathfrak{V} = -\frac{w_0^2 L^5}{B_0}\int_0^1 W(z)Y(z)\,dz,$$

—dashes denoting differentiations with respect to z. Our problem is to calculate (approximately) the configuration in which

$$\tfrac{1}{2}\mathfrak{Q} = \frac{B_0}{w_0^2 L^5}(\mathfrak{U}+\mathfrak{V}) = \tfrac{1}{2}\int_0^1 \beta(z)\{Y''(z)\}^2\,dz - \int_0^1 W(z)Y(z)\,dz \quad (20)$$

assumes its minimum value, $\beta(z)$ and $W(z)$ having forms defined by (17) and (18).

204. According to (3) of § 196

$$840\int_0^1 W(z)Y(z)\,dz \approx 41(W_0Y_0+W_6Y_6)+216(W_1Y_1+W_5Y_5)+$$
$$+27(W_2Y_2+W_4Y_4)+272W_3Y_3, \quad (21)$$

where $W_0, W_1,\ldots, W_6$ and $Y_0, Y_1,\ldots, Y_6$ stand for the values of $W(z)$ and $Y(z)$ when $z = 0, 1/6,\ldots, 6/6$ respectively. According to (18)

$$36(W_0, W_1,\ldots, W_6) = 36,\ 34,\ 28,\ 18,\ 4,\ -14,\ -36. \quad (22)$$

For the other definite integral in (20) we have, corresponding with (21),

$$840\int_0^1 \beta(z)\{Y''(z)\}^2\,dz \approx 41(\beta_0 Y_0''^2+\beta_6 Y_6''^2)+$$
$$+216(\beta_1 Y_1''^2+\beta_5 Y_5''^2)+27(\beta_2 Y_2''^2+\beta_4 Y_4''^2)+272\beta_3 Y_3''^2, \quad (23)$$

where by (17)

$$72(\beta_0, \beta_1,\ldots, \beta_6) = 72,\ 71,\ 68,\ 63,\ 56,\ 47,\ 36; \quad (24)$$

and according to (7) and Table XLI (§ 198) we have ($h = 1/6$)

$$\left.\begin{aligned}
10Y_0'' &= 1624Y_0-6264Y_1+10530Y_2-10160Y_3+5940Y_4-1944Y_5+274Y_6,\\
10Y_1'' &= 274Y_0-294Y_1-510Y_2+940Y_3-570Y_4+186Y_5-26Y_6,\\
10Y_2'' &= -26Y_0+456Y_1-840Y_2+400Y_3+30Y_4-24Y_5+4Y_6,\\
10Y_3'' &= 4Y_0-54Y_1+540Y_2-980Y_3+540Y_4-54Y_5+4Y_6,\\
10Y_4'' &= 4Y_0-24Y_1+30Y_2+400Y_3-840Y_4+456Y_5-26Y_6,\\
10Y_5'' &= -26Y_0+186Y_1-570Y_2+940Y_3-510Y_4-294Y_5+274Y_6,\\
10Y_6'' &= 274Y_0-1944Y_1+5940Y_2-10160Y_3+10530Y_4-6264Y_5+1624Y_6.
\end{aligned}\right\} \quad (25)$$

Substituting in (23) from (24) and (25), we shall obtain the wanted approximation to the definite integral in terms of $Y_0, Y_1, \ldots, Y_6$. First, however, we must take account of the terminal conditions (16).

205. According to (6) and Table XL (§ 198) we have

$$\left.\begin{aligned} 120Y'_0 &= -1764Y_0+4320Y_1-5400Y_2+4800Y_3-2700Y_4+864Y_5-120Y_6, \\ 120Y' &= 120Y_0-864Y_1+2700Y_2-4800Y_3+5400Y_4-4320Y_5+1764Y_6, \end{aligned}\right\} \qquad (26)$$

since again $h = 1/6$. The conditions (16) require that

$$\left.\begin{aligned} Y_0 = Y_6 &= 0, \\ Y'_0 = Y'_6 &= 0, \end{aligned}\right\} \qquad (27)$$

so we have from (26)

$$\left.\begin{aligned} 360Y_1-450Y_2+400Y_3-225Y_4+72Y_5 &= 0, \\ -72Y_1+225Y_2-400Y_3+450_4-360Y_5 &= 0, \end{aligned}\right\}$$

or

$$\left.\begin{aligned} 1{,}728Y_1-2{,}025Y_2+1{,}600Y_3-675Y_4 &= 0, \\ 675Y_2-1{,}600Y_3+2{,}025Y_4-1{,}728Y_5 &= 0, \end{aligned}\right\} \qquad (28)$$

as relations additional to the first of (27).

Using the four relations (27) and (28) we can reduce the number of unknown Y's from 7 to 3 (thus confirming the prediction of § 195, since in equation (2) of that section, for this example, $n = 6$ and $k = 4$). We obtain in place of (21)

$$8\times 840\int\limits_0^1 W(z)Y(z)\,dz \approx (2{,}025W_1+216W_2+675W_5)Y_2-$$
$$-\{1{,}600(W_1+W_5)-2{,}176W_3\}Y_3+(675W_1+216W_4+2{,}025W_5)Y_4,$$

W_1, W_2, W_3, W_4, W_5 having the values given in (22); that is

$$36\times 840\int\limits_0^1 W(z)Y(z)\,dz \approx 8{,}181Y_2+896Y_3-567Y_4,$$

or
$$\int\limits_0^1 W(z)Y(z)\,dz \approx 0{\cdot}270536Y_2+0{\cdot}0\dot{2}9\dot{6}Y_3-0{\cdot}01875Y_4. \qquad (29)$$

Similarly in place of (25) we obtain

$$\left.\begin{aligned} 10Y''_0 &= 2{,}430Y_2-2{,}560Y_3+1{,}215Y_4, \\ 80Y''_1 &= -6{,}255Y_2+8{,}320Y_3-3{,}735Y_4, \\ 10Y''_2 &= -315Y_2-0Y_3+180Y_4, \\ 80Y''_3 &= 3{,}645Y_2-7{,}040Y_3+3{,}645Y_4, \\ 10Y''_4 &= 180Y_2-0Y_3-315Y_4, \\ 80Y''_5 &= -3{,}735Y_2+8{,}320Y_3-6{,}255Y_4, \\ 10Y''_6 &= 1{,}215Y_2-2{,}560Y_3+2{,}430Y_4, \end{aligned}\right\} \qquad (30)$$

therefore in place of (23)†

$$840 \int_0^1 \beta(z)\{Y''(z)\}^2\,dz \approx 4{,}859{,}286{\cdot}1875Y_2^2+9{,}702{,}400Y_3^2+$$
$$+3{,}655{,}184{\cdot}1875Y_4^2-11{,}371{,}104Y_3Y_4+$$
$$+7{,}151{,}398{\cdot}875Y_4Y_2-13{,}118{,}112Y_2Y_3,$$

or

$$\int_0^1 \beta(z)\{Y''(z)\}^2\,dz \approx 5{,}784{\cdot}864Y_2^2+11{,}550{\cdot}476Y_3^2+4{,}363{\cdot}3145Y_4^2-$$
$$-13{,}537{\cdot}028Y_3Y_4+8{,}513{\cdot}570Y_4Y_2-15{,}616{\cdot}800Y_2Y_3. \quad (31)$$

206. Reverting to (20) of § 203, we have now to determine those values of Y_2, Y_3, Y_4 for which as given by (29) and (31) the quantity

$$\mathfrak{Q} = \int_0^1 \beta(z)\{Y''(z)\}^2\,dz - 2\int_0^1 W(z)Y(z)\,dz \quad (32)$$

assumes its minimum value. Inserting these values in (28) we shall obtain Y_1 and Y_5: then, since $Y_0 = Y_6 = 0$ according to the first of (27), we shall have solved our simplified problem.

The conditions for a minimum value of $\mathfrak{Q}$ are

$$\left.\begin{aligned}
0 &= \frac{\partial\mathfrak{Q}}{\partial Y_2} = 11{,}569{\cdot}729Y_2-15{,}616{\cdot}800Y_3+8{,}513{\cdot}570Y_4-0{\cdot}541072,\\
0 &= \frac{\partial\mathfrak{Q}}{\partial Y_3} = -15{,}616{\cdot}800Y_2+23{,}100{\cdot}952Y_3-13{,}537{\cdot}028Y_4-\\
&\qquad\qquad -0{\cdot}059\dot{2}5\dot{9},\\
0 &= \frac{\partial\mathfrak{Q}}{\partial Y_4} = +8{,}513{\cdot}570Y_2-13{,}537{\cdot}028Y_3+8{,}726{\cdot}629Y_4+\\
&\qquad\qquad +0{\cdot}037500,
\end{aligned}\right\} \quad (33)$$

and these may be solved either by relaxation methods or directly. The result is

$$\left.\begin{aligned}
&Y_2 = +0{\cdot}00115759,\quad Y_3 = +0{\cdot}00132796,\quad Y_4 = +0{\cdot}00092634,\\
&\text{whence according to (28)}\\
&\qquad Y_1 = +0{\cdot}00048882,\qquad Y_5 = +0{\cdot}00030815.
\end{aligned}\right\} \quad (34)$$

Substituting in (30) we deduce the corresponding approximations

† In (31)–(34) the number of decimal places to which coefficients have been computed is far beyond what can be justified on physical grounds. But in practice, when calculation is effected by machine, it is less trouble to retain figures in full than to consider the approximation of every step. (Cf. § 165.)

to Y_0'', Y_1'',..., Y_6''; and combining these with (24) we have values at 0, 1, 2,..., 6 of the 'non-dimensional bending moment'

$$\mu(z) = \beta(z)Y''(z). \tag{35}$$

207. Exact results are obtainable for comparison. The governing equation (14), when we substitute in accordance with § 203, becomes

$$\frac{d^2}{dz^2}\left\{(1-\tfrac{1}{2}z^2)\frac{d^2Y}{dz^2}\right\} = 1-2z^2$$

and gives on two integrations

$$3\frac{d^2Y}{dz^2} = z^2-1+\frac{P+Qz}{2-z^2}, \tag{36}$$

where P and Q are arbitrary. This again can be integrated to give the expressions

$$\left.\begin{aligned} 3\frac{dY}{dz} &= A\log\left(1+\frac{z}{\surd 2}\right)+B\log\left(1-\frac{z}{\surd 2}\right)-z+\tfrac{1}{3}z^3, \\ 3Y(z) &= A\left\{z\log\left(1+\frac{z}{\surd 2}\right)-z+\surd 2\log\left(1+\frac{z}{\surd 2}\right)\right\}+ \\ &\quad +B\left\{z\log\left(1-\frac{z}{\surd 2}\right)-z-\surd 2\log\left(1-\frac{z}{\surd 2}\right)\right\}-\tfrac{1}{2}z^2+\tfrac{1}{12}z^4, \end{aligned}\right\} \tag{37}$$

which both vanish (as they should) when $z = 0$, without restriction on A and B.

Because they must also vanish when $z = 1$, the constants A and B must be such as to satisfy the conditions

$$\left.\begin{aligned} &A\log\frac{\surd 2+1}{\surd 2}+B\log\frac{\surd 2-1}{\surd 2}-\tfrac{2}{3} = 0, \\ &A\left\{(\surd 2+1)\log\frac{\surd 2+1}{\surd 2}-1\right\}+B\left\{(-\surd 2+1)\log\frac{\surd 2-1}{\surd 2}-1\right\}-\tfrac{5}{12} = 0. \end{aligned}\right\} \tag{38}$$

These give $\qquad A = 1{\cdot}943722, \qquad B = 0{\cdot}303625, \qquad$ (39)

and then the expressions (37) are definite, also the function

$$\mu(z) = \beta(z)\frac{d^2Y}{dz^2} = \tfrac{1}{6}[(A-B)\surd 2-(A+B)z-2+3z^2-z^4] \tag{40}$$

which determines the bending moment in the beam.

208. In Table XLII exact values of Y_0, Y_1,..., Y_6 and of μ_0, μ_1,..., μ_6 (i.e. of $Y(z)$ and $\mu(z)$ as calculated from (37) and (40) for the values $6z = 0$, 1, 2,..., 6) are compared with their approximate values as given in (34) and as calculated from (24), (40) and (38).

It will be seen that our method has determined deflexions (i.e. the Y's) with sufficient accuracy for all practical purposes, but that sensible (though not really serious) discrepancies have resulted in the μ's (i.e. in the determination of bending moments, which are quantities of more importance in design).

TABLE XLII

	$Y(z)$		$\mu(z) = \beta(z)Y''(z)$	
$6z =$	*Exact*	*Approximate*	*Exact*	*Approximate*
0	0	0	+0·053241	+0·053888
1	0·0004825	0·0004888	+0·004575	+0·004290
2	$0{\cdot}0011521_5$	0·0011576	−0·018113	−0·018690
3	$0{\cdot}0013372_5$	$0{\cdot}0013279_5$	−0·019455	−0·019172
4	0·0009416	$0{\cdot}0009263_5$	−0·007164	−0·006489
5	0·0003112	$0{\cdot}0003081_5$	+0·007956	+0·007594
6	0	0	+0·012017	+0·012896

Alternative methods of approximation: (1) Finite-difference methods

209. The same example has been solved approximately (Ref. 4) by two alternative methods quite distinct from that of §§ 201–6: first, by a more or less conventional use of the Calculus of Finite Differences, and secondly, by relaxation methods applied on the basis of a simplifying assumption. The first method (Ref. 4, §§ 11–21) calls for only cursory notice here, since its principles are not novel and its application does not entail the relaxation technique. Its use, moreover, is restricted to governing equations of a somewhat special kind.

The formula (7) of § 198 permits us to express d^2y/dx^2 (approximately) in terms of the values assumed by y at seven equidistant points. With less close approximation (giving n in § 197 the value 2 instead of 6) we may write

$$h^2\left(\frac{d^2y}{dx^2}\right)_r \approx y_{r-1}-2y_r+y_{r+1}, \tag{41}$$

and on that understanding an exact *differential* equation of the form

$$\frac{d^2\mu}{dz^2} = W(z), \tag{42}$$

to be satisfied at every point in the range $0 \leqslant z \leqslant 1$, can be replaced by a system of *difference* equations typified by

$$N^2(\mu_{k-1}-2\mu_k+\mu_{k+1}) = W_k, \tag{43}$$

to be satisfied by $(N+1)$ quantities $\mu_0, \mu_1, \mu_2, \ldots, \mu_N$ which are the values assumed by μ at points dividing the range into N equal parts.

Given W, and *assuming* values for μ_0 and μ_1, we can use the $(N-1)$ equations of type (43) to calculate $\mu_2, \mu_3, \ldots, \mu_N$ *directly*; and then, by the same procedure applied to

$$\beta \frac{d^2 Y}{dz^2} = \mu(z), \tag{44}$$

we can arrive at an approximate solution of

$$\frac{d^2}{dz^2}\left(\beta \frac{d^2 Y}{dz^2}\right) = W(z), \tag{45}$$

which is the form assumed by (14) when we introduce the 'non-dimensional' notation of § 203. Questions will be presented by terminal conditions of the type of (16), which have to be translated into relations between $\mu_0, \mu_1, \ldots, \mu_N$ or $Y_0, Y_1, \ldots, Y_N$; but the relevant approximations (of the type of formula (6), § 198) need not concern us here. Of greater interest is the fact that errors entailed by the use of finite-difference methods in relation to (42) will be neutralized if an appropriate modification is made *initially* in the right-hand side. We have only to replace $W(z)$ by

$$W'(z) = 2\left[\frac{1}{2!}W + \frac{1}{4!\,N^2}\frac{d^2 W}{dz^2} + \frac{1}{6!\,N^4}\frac{d^4 W}{dz^4} + \ldots\right] \tag{46}$$

to make the resulting estimate of $\mu(z)$ *exact* (cf. Ref. 4, § 13).

210. What is, on the other hand, relevant to this discussion is the fact that in most engineering problems the terminal conditions are not imposed at one end only, but at both ends. This fact is exemplified in (16): it means that in the finite-difference equations we cannot start with values completely specified, but must treat some quantities as unknowns. (For example, in dealing with (42) and (44) we shall be given only two out of the four quantities μ_0, μ_1, Y_0, Y_1.) The difficulty can be surmounted when the governing equation is linear, since on that understanding solutions may be superposed.

For example, if in relation to (42) and (44) we are given the values of Y_0, Y_1 but not those of μ_0, μ_1, then we can proceed to obtain solutions on three different assumptions in turn:

(i) that $\mu_0 = \mu_1 = 0$, Y_0 and Y_1 have the specified values, and W has the specified distribution;

(ii) that $\mu_0 = 1$, $\mu_1 = Y_0 = Y_1 = 0$, and $W(z) = 0$;

(iii) that $\mu_0 = Y_0 = Y_1 = 0$, $\mu_1 = 1$, and $W(z) = 0$.

In each case the calculation is straightforward, so in this way we arrive at three solutions giving approximate forms for Y_0, Y_1, Y_2 defined as under:

Y_0 satisfies the governing equation for the specified transverse loading W, also the terminal conditions at one end $(x = 0)$;

Y_1, Y_2 entail no addition to the transverse loading, and do not affect the quantities which are specified in the terminal conditions at $(x = 0)$;

Y_1 and Y_2 are independent.

Evidently the function

$$Y = Y_0+\alpha Y_1+\beta Y_2$$

can, if α and β be suitably chosen, be made to satisfy both of the terminal conditions imposed at the other end ($x = l$; $z = 1$) as well as those imposed at $x = 0$, also the governing equation for the specified transverse loading: in other words, it is the wanted solution. Therefore our three partial solutions, similarly combined to satisfy all the terminal conditions *approximately*, will give the wanted approximation to Y.

Merits and disadvantages of the finite-difference method

211. Details of the necessary computations (made by K. N. E. Bradfield in relation to the problem solved in § 207) are fully described in the paper cited (Ref. 4). The method has also been applied (by Bradfield and by R. J. Atkinson: Ref. 1, §§ 6–15) to the more difficult problem of a girder which sustains end thrust as well as transverse loading, so that the governing equation (14) is replaced by

$$\frac{d^2}{dx^2}\left(B\frac{d^2y}{dx^2}\right)+P\frac{d^2y}{dx^2} = w, \tag{47}$$

and in test examples of which the exact solutions are known it has yielded results of comparable accuracy both for (14) and for (47). As stated in the first of the cited papers (Ref. 4, § 13), the device which is expressed in equation (46) is hardly necessary when the terminal conditions are those of simple support, but it is well worth while—in fact, almost essential—when the ends of the beam are clamped. In the second case there is the further complication (Ref. 4, § 17) that accurate expressions for the terminal slopes involve differences of rather high order: consequently the labour of computation is not reduced so far as might be expected when N, in equation (43), is made fairly small.

In Bradfield's work (Ref. 4) the finite-difference method was applied on the basis of the value $N = 10$. Table XLIII exhibits the results obtained in the problem of §§ 203, 207, the notation being comparable with that of Table XLII.

TABLE XLIII

$z =$	$Y(z)$		$\mu(z) = \beta(z)Y''(z)$	
	Exact	*Approximate*	*Exact*†	*Approximate*†
0	0	0	+0·053240	+0·053223
0·1	0·0002084	0·0002082	+0·020767$_5$	+0·020754
0·2	0·0006325	0·0006330	−0·001938	−0·001948
0·3	0·0010447	0·0010451	−0·015477	−0·015483
0·4	0·0013018	0·0013015	−0·020849	−0·020852
0·5	0·0013371‡	0·0013370	−0·019455	−0·019454
0·6	0·0011555	0·0011551	−0·013093$_5$	−0·013089
0·7	0·0008179	0·0008172	−0·003966	−0·003958
0·8	0·0004294	0·0004290	+0·005329	+0·005340
0·9	0·0001185	0·0001185	+0·011790	+0·011805
1	0	0	+0·012018	+0·012036

† The last figures as given by Bradfield (Ref. 4: Tables I and VIII) have here been rounded off.

‡ Bradfield's figure corrected by D. G. Christopherson.

Alternative methods of approximation: (2) The use of finite series of chosen functions

212. The second of the two alternative methods mentioned in § 209 is a direct application of relaxation methods, made possible by a simplifying assumption which also forms the basis of older methods due to Rayleigh and to Galerkin (Refs. 5 and 6). This assumption is that a wanted function, e.g. the function $Y(z)$ in the equation

$$\beta(z)\frac{d^2Y}{dz^2} = \mu(z), \tag{44 bis}$$

can be represented *with sufficient accuracy* by a finite series of the form

$$Y(z) = a_1 Y_1 + a_2 Y_2 + \ldots + a_n Y_n, \tag{48}$$

in which $Y_1, Y_2, \ldots, Y_n$ are functions of z which *severally satisfy all* of the imposed terminal conditions, are independent,† and are single-valued for every value of z in the range considered; otherwise they can be chosen arbitrarily.‡ On that understanding no restriction is imposed by the terminal conditions on the coefficients $a_1, a_2, \ldots, a_n$,

† In the sense that no one of them can be obtained from the others by synthesis.

‡ It should be emphasized that the symbols $Y_1, Y_2, \ldots$, etc., now have meanings entirely different from those which they carried in §§ 204–8.

which accordingly can be chosen to satisfy other conditions; for example, to make the curve of deflexion pass through, or satisfy the governing equation at, n specified points in the range $0 \leqslant z \leqslant 1$.

Obviously an expression of the type of (48), involving only a finite number of arbitrary coefficients, will not in general represent a wanted function *exactly* in the mathematical sense. But exact representation has little importance in practical work, and indeed is almost meaningless: all that we need is a *sufficient* representation, in the sense that a number of plotted points may be said to define sufficiently a relation sought in experiment (cf. § 194). From this standpoint it is evident that we can improve the accuracy of (48), first by increasing the number (n) of its terms, and secondly by an appropriate choice of the functions $Y_1, Y_2,..., Y_n$. It is to be expected that different classes of function will have advantages in different problems.

'Systematic Relaxation'

213. The finite series (48) can be differentiated term by term: therefore (still regarding the a's as undetermined coefficients) we can obtain corresponding series expressions for $Y'(z)$ and $Y''(z)$ and substitute these expressions in the integrals which appear in (20), viz.

$$\int_0^1 \beta(z)\{Y''(z)\}^2\,dz \quad \text{and} \quad \int_0^1 W(z)Y(z)\,dz.$$

Then, $\beta(z)$ and $W(z)$ being specified, if $Y_1, Y_2,..., Y_n$ are suitably chosen the integrals can be evaluated in the form of quadratic and linear functions, respectively, of the a's, with *known* numerical coefficients. Thus we shall obtain, finally, an expression for $(\mathfrak{U}+\mathfrak{V})$ in (20) from which n conditions of minimum total energy can be derived, namely,

$$\frac{\partial(\mathfrak{U}+\mathfrak{V})}{\partial a_1} = \frac{\partial(\mathfrak{U}+\mathfrak{V})}{\partial a_2} = ... = \frac{\partial(\mathfrak{U}+\mathfrak{V})}{\partial a_n} = 0. \tag{49}$$

These are identical in form with (15); so we are left with a problem exactly similar to that of §§ 203–8.

214. The type equation (49) may be written as

$$\frac{\partial}{\partial a_r}\left[\tfrac{1}{2}\int_0^1 \beta(z)\{Y''(z)\}^2\,dz - \int_0^1 W(z)Y(z)\,dz\right] = 0, \tag{50}$$

and when $Y(z)$ is given by

$$Y(z) = a_1 Y_1 + a_2 Y_2 + ... + a_n Y_n \tag{48) \textit{bis}$$

we have (cf. § 213)

$$Y''(z) = a_1 Y_1'' + a_2 Y_2'' + ... + a_n Y_n''; \tag{51}$$

so (50) can be written in accordance with the standard conventions of relaxation theory as follows:

$$\left.\begin{aligned}
&\mathsf{A}_r = \mathsf{A}_r + A_r = 0, \\
&\text{where} \\
&\mathsf{A}_r \text{ (the `initial force' corresponding with } a_r) \\
&\quad = -\frac{\partial}{\partial a_r}\left[-\int_0^1 W(z)Y(z)\,dz\right] = \int_0^1 W(z)Y_r\,dz, \\
&\text{and} \\
&A_r \text{ (the `force due to displacements' } a_1, a_2,..., a_n) \\
&\quad = -\frac{\partial}{\partial a_r}\left[\tfrac{1}{2}\int_0^1 \beta(z)\{Y''(z)\}^2\,dz\right] = -\int_0^1 \beta(z)Y''(z)Y_r''\,dz \\
&\quad = -\int_0^1 \beta(z)Y_r''(a_1 Y_1'' + a_2 Y_2'' + ... + a_n Y_n''\,dz).
\end{aligned}\right\} \tag{52}$$

Thus to find the 'initial forces' we must calculate (exactly or approximately) all such integrals as $\int_0^1 W(z)Y_r\,dz$; and to formulate the operations table we must similarly calculate all 'influence coefficients' of the types

$$\left.\begin{aligned}
\widehat{rr} &\equiv \frac{\partial A_r}{\partial a_r} = -\int_0^1 \beta(z)Y_r''^2\,dz \\
\text{and}\qquad \widehat{sr} = \widehat{rs} &\equiv \frac{\partial A_r}{\partial a_s} = -\int_0^1 \beta(z)Y_r''Y_s''\,dz.
\end{aligned}\right\} \tag{53}$$

The 'Maxwell relations' are satisfied, and all 'influence coefficients' of type $\widehat{rr}$ are negative; so the standard proof of convergence (Chap. V) will hold. The definite integrals can be evaluated either exactly or approximately—i.e. by graphical or numerical methods, or with the use of formulae of the type of (3), § 196.

215. Having the Operations Table we can adopt the standard relaxation procedure. Initially (all a's being zero) the A's vanish and so $\mathsf{A}_r = \mathsf{A}_r$ according to (52): values of the 'initial forces' can thus be calculated from the second of (52) and inserted in the first line of a Relaxation Table, for liquidation in the normal way. When every

residual force has been brought (sensibly) to zero we shall have values of $a_1, a_2,..., a_n$ which make the total potential energy a minimum, subject only to restrictions imposed by the form of the assumed expression (48).† If (48) permits a close representation of the wanted function $Y(z)$, then a closely approximate solution will be obtained when the computed values are substituted for $a_1, a_2,..., a_n$.

Requirements of the chosen functions $Y_1, Y_2,..., Y_n$

216. We have said (§ 212) that the accuracy of (48) depends both on the number (n) of its arbitrary coefficients and on the forms of $Y_1, Y_2,..., Y_n$; also that the most appropriate forms for these functions will depend upon the type of the problem studied. It will usually be advisable to base them on a known solution for some simpler case: e.g. the governing equation (14) of a transversely loaded girder is soluble without difficulty when the girder has uniform flexural rigidity and when the loading is sinusoidally distributed, and these solutions can be taken as giving $Y_1, Y_2,..., Y_n$.

When B is constant, equation (14) simplifies to

$$B\frac{d^4y}{dx^4} = w, \tag{14a}$$

i.e.

$$Y^{\text{iv}} = W(z) \tag{54}$$

in the 'non-dimensional' notation of § 203. Therefore if

$$W(z) = W_r \text{ (say)} = \sin r\pi z \quad (r \text{ integral}),$$

we have

$$Y(z) = Y_r \text{ (say)} \propto (\sin r\pi z + A + Cz + Fz^2 + Gz^3), \tag{55}$$

where A, C, F, G are constants of integration. This is a single-valued and differentiable function of z; by attaching suitable values to A, C, F, G it can be made to satisfy all of the imposed terminal conditions; and that the Y's for different r's are mutually independent follows from the mutual independence of the harmonic W's with which they correspond.

Forms for $Y_1, Y_2,..., Y_n$ appropriate to various terminal conditions

217. Corresponding with (55) we have

$$\left.\begin{aligned} Y' &\propto r\pi \cos r\pi z + C + 2Fz + 3Gz^2, \\ Y_r'' &\propto -r^2\pi^2 \sin r\pi z + 2F + 6Gz, \end{aligned}\right\} \tag{i}$$

† In practice $W(z)$, and therefore the initial forces, will not be known exactly, so that it is only necessary to bring the unliquidated forces within some finite margin of uncertainty.

so Y_r will satisfy conditions of simple support (viz. $Y_r = Y_r'' = 0$) at both ends ($z = 0$ and $z = 1$) if

$$A = C = F = G = 0; \tag{ii}$$

that is to say, $$Y_r = \sin r\pi z \quad (Y_r'' = -r^2\pi^2 \sin r\pi z) \tag{56}$$

is a type-solution appropriate to *simply supported ends*. Table XLIV records (with a view to approximate calculation of such quantities as $\widehat{rr}$, $\widehat{rs}$, § 214) values of Y_r at every tenth part of the range $0 \leqslant z \leqslant 1$, and for $r = 1, 2,..., 9$;† Table XLV similarly records Y_r''. The last two lines of each table record two quantities which have been found useful in computation: $\sum_0^{1\cdot 0}(Y)$ denotes the algebraic sum of the Y values, and $\sum_0^{1\cdot 0}|(Y)$ denotes the arithmetical sum,—i.e. the sum taken without regard to sign.

218. The type-solution (55) will satisfy conditions of *clamping* (viz. $Y_r = Y_r' = 0$) if

$$A = 0, \quad C = -r\pi, \quad F = r\pi(2+\cos r\pi), \quad G = -r\pi(1+\cos r\pi).$$

That is to say,

$$\left.\begin{aligned} Y_r &= \sin r\pi z + r\pi\{-z+(2+\cos r\pi)z^2-(1+\cos r\pi)z^3\}, \\ \text{giving} \quad Y_r' &= r\pi\{\cos r\pi z - 1 + 2(2+\cos r\pi)z - 3(1+\cos r\pi)z^2\}, \\ Y_r'' &= -r^2\pi^2 \sin r\pi z + 2r\pi\{2+\cos r\pi - 3(1+\cos r\pi)z\}, \end{aligned}\right\} \tag{57}$$

is a type-solution appropriate to *clamped ends*. Tables XLVI and XLVII record, for $r = 1, 2,..., 9$, values of Y_r and Y_r'' as calculated from (57) at every tenth part of the range $0 \leqslant z \leqslant 1$.

Other terminal conditions will entail other forms for the type-solutions, to be found in the same way. It should be emphasized that only the terminal conditions restrict our choice: it was in no way necessary to introduce equation (14 a), which might (for example) have been replaced by the governing equation of a uniform *vibrating* bar.

An example treated by 'Systematic Relaxation': Deflexions and bending moments in a clamped beam

219. As an example we now consider the problem stated in § 203, of a beam clamped at both ends, having a flexural rigidity defined

† Tables XLIV–XLVII, as having general application, are placed at the end of the book. They were computed by K. N. E. Bradfield (Ref. 4).

by (17), and carrying a transverse loading defined by (18). The exact solution of this problem has been given in § 207.

Type-solutions appropriate to clamped ends have been suggested in (57). Substituting that expression for Y_r and $(1-\frac{1}{2}z^2)$ for $\beta(z)$, $(1-2z^2)$ for $W(z)$ in accordance with (17) and (18), we have from (52) for the typical 'initial force'

$$\mathsf{A} = \int\limits_0^1 W(z)Y_r\,dz =$$

$$= \frac{1}{r\pi}\left[1+\cos r\pi+\frac{4}{r^2\pi^2}(1-\cos r\pi)\right]-\frac{r\pi}{60}(3-\cos r\pi), \qquad (58)$$

and from (53) as typical 'influence coefficients'

$$\widehat{rr} = -\int\limits_0^1 \beta(z)Y_r''^2\,dz$$

$$= -\left[\frac{5}{12}r^4\pi^4-\frac{r^2\pi^2}{120}(649+344\cos r\pi)-8(5+4\cos r\pi)\right], \qquad (59)$$

$$\begin{aligned}\widehat{rs} = rs\Big[&\frac{\pi^2}{30}\{128+43(\cos r\pi+\cos s\pi)+38\cos r\pi\cos s\pi\}+\\ &+\frac{2r^2s^2\pi^2}{(r^2-s^2)^2}\cos r\pi\cos s\pi+\\ &+\frac{2}{s^2}(2+\cos r\pi+7\cos s\pi+8\cos r\pi\cos s\pi)+\\ &+\frac{2}{r^2}(2+7\cos r\pi+\cos s\pi+8\cos r\pi\cos s\pi)\Big]. \end{aligned} \qquad (60)$$

Substituting the integral values 1, 2,..., 9 in turn for r, we obtain from (58) nine component 'forces' as under:

$$\left.\begin{aligned} \mathsf{A}_1 &= 0\cdot0485727, & \mathsf{A}_2 &= 0\cdot1088702, & \mathsf{A}_3 &= -0\cdot6187628,\\ \mathsf{A}_4 &= -0\cdot2597242, & \mathsf{A}_5 &= -1\cdot0451339, & \mathsf{A}_6 &= -0\cdot5222154,\\ \mathsf{A}_7 &= -1\cdot4653250, & \mathsf{A}_8 &= -0\cdot7581810, & \mathsf{A}_9 &= -1\cdot8846025. \end{aligned}\right\} \qquad (61)$$

Substituting in the same way for r and for s in (59) and (60) we obtain the Operations Table XLVIII.

220. We can now start the relaxation process, taking any number n ($\ngtr 9$) of the initial forces (61) and retaining the same number of rows and columns in Table XLVIII. Naturally the approximation of the results will increase with n; but even when we intend to use the full range ($n = 9$) of our calculations, preliminary estimates made

TABLE XLVIII. *Values of $\widehat{rr}$ and $\widehat{rs}$ according to* (59) *and* (60)

$s \backslash r$	1	2	3	4	5	6	7	8	9
1	−7·501877	−6·328337	+100·617647	+16·820545	+156·678408	+30·172333	+215·742651	+42·324153	+275·563370
2	−6·328337	−250·710030	−8·893885	+823·421269	+241·728022	+1,148·166038	+380·918396	+1,501·933664	+508·209035
3	+100·617647	−8·893885	−3,053·789600	−372·803483	+664·084552	+327·044967	+677·998474	+556·404964	+798·892158
4	+16·820545	+823·421269	−372·803483	−9,011·567354	−1,376·575223	+2,749·899303	+417·430637	+3,023·685076	+837·273679
5	+156·678408	+241·728022	+664·084552	−1,376·575223	−24,731·819512	−3,545·152444	+2,398·923305	+315·374380	+1,767·351090
6	+30·172333	+1,148·166038	+327·044964	+2,749·899303	−3,545·152444	−49,588·753666	−7,430·483532	+6,838·861458	+47·955034
7	+215·742651	+380·918396	+677·998474	+417·430637	+2,398·923305	−7,430·483532	−96,212·500540	−13,776·094555	+6,486·396279
8	+42·324153	+1,501·933664	+556·404964	+3,023·685076	+315·374380	+6,838·861458	−13,776·094555	−160,945·905126	−23,382·994306
9	+275·563370	+508·209035	+798·892158	+837·273679	+1,767·351090	+47·955034	+6,486·396279	−23,382·994307	−264,252·196076

TABLE XLIX. *Progress of Convergence with an increasing number of Chosen Functions*

Number of available parameters	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1	0·006475								
2	0·006240	0·0002769							
3	0·006296	0·0002752	0·000004						
4	0·006248	0·00031437	0·000000918	0·000011528					
5	0·0062345	0·00031464	0·000000396	0·000011598	−0·000000322				
6	0·006220	0·0003278	−0·0000002	0·00001335	−0·00000063	0·000001627			
7	0·006212	0·0003278	−0·0000005	0·00001335	−0·0000007	0·000001639	−0·000000092		
8	0·006196	0·0003343	−0·00000112	0·00001416	−0·00000084	0·000001899	−0·000000181	0·0000003952	
9	0·006192	0·0003345	−0·00000127	0·00001419	−0·000000875	0·000001908	−0·000000197	0·0000004022	−0·000000036

with smaller n's will help to give good starting assumptions. Thus A_1, taken singly, can be liquidated by a 'displacement'

$$a_1 = -A_1/\widehat{11}$$

in the first approximation ($n = 1$). Substituting from (61) and Table XLVIII we deduce the value

$$a_1 = \frac{0{\cdot}048572}{7{\cdot}50187} = 0{\cdot}006475,$$

as a starting assumption for the second approximation ($n = 2$).

Proceeding in this way with successively increased values of n, we obtain the sequence of approximations given in Table XLIX, which serves to demonstrate the rapidity of the convergence obtainable by this method. Having computed values of the a's in the series (48), to complete the solution we may calculate $Y(z)$ and $\beta(z)Y''(z)$ for a number of points in the range $0 \leqslant z \leqslant 1$, deriving these quantities by synthesis in accordance with (48) of values taken from the relevant tables of $Y_1, Y_2, \ldots, Y_n$ and $Y_1'', Y_2'', \ldots, Y_n''$ (in this instance Tables XLVI and XLVII). Table L compares with exact calculations (based on § 207) results obtained in this manner from the last line of Table XLIX. The agreement is very close.

TABLE L

$z =$	$Y(z)$		$\mu(z) = \beta(z)Y''(z)$	
	Exact	*Approximate*	*Exact*	*Approximate*
0	0	0	+0·053240	+0·052800
0·1	0·0002084	0·0002081	+0·020767$_5$	+0·020726
0·2	0·0006325	0·0006327$_5$	−0·001938	−0·001901$_5$
0·3	0·0010447	0·0010449	−0·015477	−0·015498$_5$
0·4	0·0013018	0·0013014$_5$	−0·020849	−0·020838$_5$
0·5	0·0013371	0·0013371	−0·019455	−0·019453$_5$
0·6	0·0011555	0·0011553$_5$	−0·013093$_5$	−0·013104
0·7	0·0008179	0·0008176	−0·003966	−0·003947$_5$
0·8	0·0004294	0·0004294$_5$	+0·005329	+0·005293
0·9	0·0001185	0·0001187	+0·011790	+0·011825
1	0	0	+0·012018	+0·012443

The method of 'Relaxation by Inspection'

221. The method described in §§ 212–18 and exemplified in §§ 219–20 is accurate and very powerful, but in the construction of the operations table it calls for integrations which may be troublesome and in any event demand more of the computer than the mere arithmetic which is entailed by the relaxation process. On this

account when very high accuracy is not essential (as is the rule in practical work) a more pedestrian method may be preferable which Bradfield (Ref. 3) has termed 'Relaxation by Inspection'. (It is the 'Second Method' of Ref. 4.)

This method too entails the assumption stated in (48) and calls for a choice of type-solutions appropriate to the terminal conditions. But it dispenses with the integral formulae of § 214, and it starts by deriving (either exactly or by relaxation methods) the bending moment μ_S which corresponds according to equation (42) with the specified loading $W(z)$, *and which vanishes at both ends of the range.* Thus when $W(z)$ is given by (18), as in the numerical example of this chapter, then μ_S is given by†

$$\mu_S = -\tfrac{1}{6}z(1-z)^2(2+z). \tag{62}$$

Values of μ_S for the points

$$z = 0,\ 0{\cdot}1,\ 0{\cdot}2,\ \ldots,\ 0{\cdot}9,\ 1$$

are inserted in the first line of a relaxation table.

Our problem now is to satisfy the equation

$$\beta(z)Y''(z) = \mu_C(z) \tag{44 bis}$$

$$= \mu_S+\alpha+\gamma z \tag{63}$$

by Statics,‡ α and γ being constants (for the moment unknown) which appear in virtue of the terminal moments due to clamping; and we seek an approximate solution in which (44) is satisfied not everywhere but at a number of points in the range $0 \leqslant z \leqslant 1$, representing $Y(z)$ for this purpose by a finite series (48) of chosen functions, each of which satisfies the imposed terminal conditions.

222. Now if μ_r be defined by the equation

$$\left.\begin{array}{l} \mu_r = \beta Y_r''-\alpha_r-\gamma_r z \\ \text{combined with the conditions} \\ \mu_r = 0 \quad \text{when } z = 0 \text{ and when } z = 1, \end{array}\right\} \tag{64}$$

then we have by (51) and (63)

$$\begin{aligned} \mu_S+\alpha+\gamma z &= \beta(z)Y''(z) \\ &= \beta(z)\{a_1 Y_1''+a_2 Y_2''+\ldots+a_n Y_n''\} \\ &= \sum_n [a_r(\mu_r+\alpha_r+\gamma_r z)], \text{ by (64).} \end{aligned}$$

† Cf. Ref. 4, equation (15). ‡ Cf. Ref. 4, equation (10).

Since μ_S and $\mu_1, \mu_2, \ldots, \mu_n$ vanish severally at both ends of the range, this means that

$$\left.\begin{aligned} \mu_S &= \sum_n [a_r(\mu_r)], \\ \alpha &= \sum_n [a_r \alpha_r], \\ \gamma &= \sum_n [a_r \gamma_r]. \end{aligned}\right\} \tag{65}$$

If then we satisfy the first of (65) within the limits permitted by our starting assumption (i.e. if we satisfy it at n points which subdivide the range into $(n+1)$ parts), we shall determine the n coefficients of type a_r and therefore (α_r, γ_r being determined by the terminal conditions imposed on μ_r) we can calculate α and γ from the second and third of (65). Then μ_C will be known from (63) and (because the a's have been determined) we can also deduce a definite approximation to $Y(z)$.

A numerical example: Deflexions and bending moments in a clamped beam

223. We have explained the method in relation to the problem of a beam with clamped ends, and we shall illustrate it by the same example as before. Type-solutions appropriate to clamped ends have had values tabulated in Tables XLVI and XLVII for a tenfold subdivision of the range. Using Table XLVII and values of $\beta(z)$ calculated from (17), we deduce values of $\beta Y_r''$ by multiplication and thence derive an Operations Table (Table LI) giving corresponding values of μ_r as defined in (64). This we use in accordance with the standard relaxation technique to liquidate μ_S as calculated from (62), judging at every stage (with the aid if necessary of a rough plotting) the appropriate operator and multiplier to be used next.

In Table LI we also record the terminal values of $\beta Y_r''$ (i.e. of $\alpha_r + \gamma_r z$), for use in the final synthesis represented by (65). That synthesis, as we have shown, gives the required solution μ_C, and it can be seen from (65) that if only terminal values of μ_C are required they may be obtained from the equation

$$\mu_C = \sum [A_r(\alpha_r + \gamma_r z)], \quad \text{when } z = 0 \text{ or } 1, \tag{66}$$

since μ_S vanishes at either end.

Table LII summarizes the results of the relaxation process,† Table LIII presents the final approximations to $Y(z)$ and μ_C $(= \beta(z)Y'')$.

† The actual relaxation (not reproduced) involved twenty-five operations. The criterion of liquidation was taken to be the quantity $\sum (\mu)$ calculated without regard to sign, and in twenty-five operations this quantity was reduced from 3305·4 (the initial value of $\sum (\mu_S)$) to 3·8.

Table LI

Section no.	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
0	0	0	0	0	0	0	0	0	0
1	−2·75188	−25·12455	−70·65456	−153·50540	−244·09270	−342·33353	−387·31377	−377·56396	−243·26004
2	−5·18252	−41·01759	−81·28141	−99·40754	+2·51326	+191·99072	+454·26078	+571·83531	+465·02381
3	−6·96562	−42·19008	−24·23448	+75·97556	+238·93547	+180·44518	−138·10079	−599·03983	−611·71715
4	−7·88163	−29·49145	+50·29595	+121·88401	+3·76991	−335·31161	−256·24060	+309·00312	+706·27164
5	−7·85050	−9·42478	+80·07933	−18·84056	−211·97062	−28·27434	+428·65708	−37·69911	−692·43964
6	−6·94299	+9·07540	+45·07487	−143·05668	+3·76990	+247·23330	−227·81468	−344·25768	+630·24057
7	−5·36869	+18·84722	−18·74470	−89·07895	+189·58745	−186·17736	−108·21201	+415·55747	−482·36536
8	−3·44216	+17·69001	−55·93773	+47·43431	+2·51327	−165·53782	+316·27848	−439·86838	+324·05444
9	−1·53193	+9·05679	−41·90972	+79·85079	−145·39666	+186·80967	−230·81408	+201·90967	−144·44406
10	0	0	0	0	0	0	0	0	0
$\sum(\mu_r)$	−47·91792	−92·57903	−117·31245	−178·74426	−100·77072	−251·14690	−149·29759	−298·12339	−48·63579
$(\alpha_r+\gamma_r z)_0$	+6·28319	+37·69911	+18·84956	+75·39822	+31·41593	+113·09734	+43·98230	+150·79645	+56·54867
$(\alpha_r+\gamma_r z)_{10}$	+3·14159	−18·84956	+9·42478	−37·69911	+15·70797	−56·54867	+21·99115	−75·39823	+28·27434

Table LII. *Summary of Relaxation Process for determining $\mu_C \times 10^4$*

Operation and multiplier	0	1	2	3	4	5	6	7	8	9	10
Operation 1 × 61·70	0	−169·8	−319·8	−429·8	−486·3	−484·4	−428·4	−331·2	−212·4	−94·5	0
,, 2 × 3·45	0	−86·7	−141·5	−145·6	−101·7	−32·5	31·3	65·0	61·0	31·2	0
,, 3 × −0·02	0	1·4	1·6	0·5	−1·0	−1·6	−0·9	0·4	1·1	0·8	0
,, 4 × 0·153	0	−23·5	−15·2	11·6	18·6	−2·9	−21·9	−13·6	7·3	12·2	0
,, 5 × −0·01	0	2·4	0	−2·4	0	2·1	0	−1·9	0	1·5	0
,, 6 × 0·02	0	−6·8	3·8	3·6	−6·7	−0·6	4·9	−3·7	−3·3	3·7	0
,, 7 × −0·002	0	0·8	−0·9	0·3	0·5	−0·9	0·5	0·2	−0·6	0·5	0
,, 8 × 0·004	0	−1·5	2·3	−2·4	1·6	−0·1	−1·4	1·7	−1·8	0·8	0
Sum	0	−283·7	−469·7	−564·2	−575·0	−520·9	−415·9	−283·1	−148·7	−43·8	0
μ_S (specified) $\times 10^4$	0	−283·5	−469·3	−563·5	−576·0	−520·8	−416·0	−283·5	−149·3	−43·5	0
Remainder unliquidated	0	+0·2	+0·4	+0·7	−1·0	+0·1	−0·1	−0·4	−0·6	+0·3	0

Comparing these results for Y_C, μ_C with the accurate values given in the first and third columns of Table XLIII, we see that very satisfactory accuracy has been attained in this example by the 'Method of Relaxation by Inspection'. Only three values are in error by as much as 0·1 per cent., and none by 0·5 per cent.

TABLE LIII. *μ_C and Y_C as determined 'by inspection'* (*Bradfield: Ref.* 4, *Table XIII*)

Section no.	$(\alpha+\gamma z)\times 10^2$	$\mu_S\times 10^2$	$\mu_C\times 10^2$	$Y_C\times 10^4$
0	5·3134	0	5·3134	0
1	4·9033	−2·8350	2·0683	2·084
2	4·4932	−4·6933	−0·2001	6·325
3	4·0831	−5·6350	−1·5519	10·447
4	3·6730	−5·7600	−2·0870	13·018
5	3·2629	−5·2083	−1·9454	13·368
6	2·8527	−4·1600	−1·3073	11·536
7	2·4426	−2·8350	−0·3924	8·179
8	2·0325	−1·4933	+0·5392	4·289
9	1·6224	−0·4350	1·1874	1·180
10	1·2123	0	1·2123	0

RECAPITULATION

224. This like the preceding chapter leads from work already published to developments now presented for the first time. The computational work for the new sections (§§ 203–8 and 219–20) has been done by Mr. G. C. J. Dalton and Mr. F. S. Shaw; the remainder of the chapter (excepting §§ 196–200, for which I am indebted to Dr. W. G. Bickley) has been taken in the main from papers written in collaboration with R. J. Atkinson and K. N. E. Bradfield (Refs. 1 and 4).

Four alternative methods are propounded, all illustrated by the same example (§ 203). It is too early as yet to pronounce on their relative merits: probably each will prove to have its own advantages for research. Briefly, the aim of this chapter is to bring within range of relaxation methods systems which in fact have infinite freedom; that is, at some cost in accuracy to reduce the freedom of a given system to some finite order N. By the first method as applied here (§§ 196–206) N is given a value $(7-k)$ where k is the number of the imposed terminal conditions; consequently in the flexural example (§ 203), where $k = 4$, the order of the freedom is reduced to only 3. In view of this drastic simplification the accuracy attained in the solution (Table XLII) may be deemed satisfactory, and it will

often happen that the experimental data (e.g. the measured stiffness of an aeroplane wing) do not justify more elaborate treatment. The other methods make less drastic simplification, and as was to be expected they yield closer approximations to the 'correct' result.

The second method (§§ 209–11) is the oldest of the four: it has been described very fully in Ref. 4, and having no outstanding advantage is merely summarized here. The third and fourth have a different basis (§ 212), the third (§§ 213–15 and 219–20) being a recent and more systematic development of a method used in Refs. 3 and 4 (§§ 221–2). In this connexion Table XLIX has considerable interest, as showing the rapidity of the convergence which in either of these two (fundamentally equivalent) methods results from an increase in the number (N) of available parameters. The accuracy attainable with either seems to be limited only by considerations of time and labour.

225. This may be made occasion for a remark on the 'effective accuracy' of exact solutions such as that of § 207. In theory (when, as here, the governing equation is integrable) the solution presented in (37) and (38) is exact; but in practice, when a functional solution of this kind has to be *computed*, accuracy may be very hard to attain because the wanted quantity appears as a small difference of two large quantities. Referring to Tables XLII and XLIII, the reader will notice that $\mu(z)$ at the end points ($z = 0$, $z = 1$) is given there by figures which differ by one digit in the last place,—that is, by approximately 1 in 50,000 and 1 in 12,000. The discrepancy has been traced to the use of two figures for $\sqrt{2}$ which differed by one digit in the *eighth* significant figure,—i.e. by less than 1 in 10^7.

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XI

CONTINUOUS SYSTEMS. II. ELASTIC STABILITY AND VIBRATIONS

226. THE methods of Chapter X, since they can be used to express **T** and **V** (approximately) in terms of a finite number of coordinates, bring within the scope of Chapters VII and VIII all problems relating to the torsional or flexural vibrations of continuous bars, also an allied class of problems which relates to Elastic Stability. Similar methods applied to the integral expression for the Dissipation Function $\mathfrak{F}$ (§ 172) permit an approximate allowance for damping forces in the manner of Chapter IX, and an obvious and simple extension of the theory will take account of forced oscillations. These matters form the material of the present chapter, which so far widens the scope of relaxation methods as to give it promise of being useful in fields still largely unexplored.

Elastic stability. General theory

227. In principle the theory of elastic stability has much in common with the theory of vibrations; but in practice there is this difference, that hardly any question of elastic stability arises in relation to systems of finite freedom. On that account we have had as yet no occasion to develop the principles of stability theory, and a brief review (to be read in parallel with §§ 126–33) will form a fitting introduction to this chapter.

In Chapter VII $\mathfrak{V}$ was throughout assumed to be an *essentially positive* function of the coordinates. This will be true of practical systems provided that $\mathfrak{V}$ consists wholly of elastic strain-energy:† when on the other hand external forces are operative, then (in certain circumstances) $\mathfrak{V}$ may be zero or negative. Consider for example a uniform strut subjected to end thrust P: the elastic strain-energy of flexure is given in terms of the deflexion y by

$$\mathfrak{V}_1 = \frac{1}{2}\int_0^l B\left(\frac{d^2y}{dx^2}\right)^2 dx, \tag{1}$$

an essentially positive quantity, since B (the flexural rigidity) cannot be negative; but a further contribution to the potential energy comes

† *Elasticity* § 480.

from the applied forces, and because in virtue of its curvature the two ends of the central line approach one another, relatively to the straight configuration this second part $\mathfrak{V}_2$ is negative. We have†

$$\mathfrak{V}_2 = -\tfrac{1}{2}P\int_0^l \left(\frac{dy}{dx}\right)^2 dx, \tag{2}$$

so according to the magnitude of P the change due to flexure of the total potential energy,—namely,

$$\mathfrak{V} = \mathfrak{V}_1+\mathfrak{V}_2, \tag{3}$$

—may be *either positive or negative.*

If $\mathfrak{V}$ is positive for all permitted forms of deflexion, so that the total potential energy is a minimum in the straight configuration, then according to statical principles that configuration is stable. If on the other hand for any permitted form of deflexion $\mathfrak{V}$ can be negative, then by assuming this form the strut can pass to a position of lower potential energy, therefore is unstable. Evidently conditions of neutral or limiting stability will be attained when for some permitted form of y

$$2\mathfrak{V} = \int_0^l B\left(\frac{d^2y}{dx^2}\right)^2 dx - P\int_0 \left(\frac{dy}{dx}\right)^2 dx = 0, \tag{4}$$

and the problem confronting the designer is—What is the least value of P for which (4) possesses a solution other than the obvious (and nugatory) solution $y = 0$?

228. If (4) be compared with the similarly numbered equation of Chapter VII, it is apparent that this is a problem exactly analogous to what is standard in vibration theory. For whereas in that theory we have, given the forms of $\mathfrak{V}$ and $\mathfrak{T}$, to deduce 'characteristic values' of p^2 and the mode appropriate to each, here we are given the forms of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ in the equation

$$\mathfrak{V}_1 = P.\mathfrak{V}_2 \tag{5}$$

and have to deduce characteristic values of P. (In fact the present problem is more simple, in that from a practical standpoint only the smallest value of P is significant, whereas in vibration problems it is often necessary to determine frequencies higher than the gravest.)

With appropriate forms for $\mathfrak{V}_1$ and $\mathfrak{V}_2$, every problem relating to

† *Elasticity* § 480.

elastic stability can be expressed in the form of (5), therefore comes within range of the methods already propounded for vibration problems. Here, for simplicity, we shall confine attention to the stability of compression members (struts); but clearly no alteration in the processes of computation will be required to render them applicable to more complex problems.

A numerical example (1): Straight compression member of non-uniform rigidity

229. According to § 227, in this instance we have

$$2\mathfrak{V}_1 = \int_0^l B\left(\frac{d^2y}{dx^2}\right)^2 dx, \qquad 2\mathfrak{V}_2 = \int_0^l \left(\frac{dy}{dx}\right)^2 dx \tag{6}$$

in the 'energy equation' (5). To express the problem in 'non-dimensional' terms we shall write

$$z = x/l, \qquad \beta(z) = B/B_0, \qquad \lambda = Pl^2/B_0, \tag{7}$$

B_0 denoting the value of B at some specified (e.g. the central) section. Then (5) and (6) can be replaced by

$$\left.\begin{aligned} &\mathbf{V} = \lambda\mathbf{V}', \\ \text{where} \qquad &2\mathbf{V} = \int_0^1 \beta(z)y''^2\,dz, \qquad 2\mathbf{V}' = \int_0^1 y'^2\,dz, \end{aligned}\right\} \tag{8}$$

dashes denoting differentiations with respect to z.

As in preceding chapters we take for numerical treatment an example of which the exact solution is known. A suitable example is provided by the case in which

$$\beta(z) = (1-4k^2z^2)^2 \tag{9}$$

when the origin is taken at the middle point of the strut. B_0 relating to the central section, the exact solution for simply supported ends gives for the characteristic numbers

$$\lambda = 4k^2\left[1+\frac{n^2\pi^2}{\left\{\log_e\left(\frac{1+k}{1-k}\right)\right\}^2}\right], \tag{10}$$

n having any integral value. In particular, when $(1-k^2)^2 = 0{\cdot}2$ the first four values of λ (corresponding with $n = 1, 2, 3, 4$) are 8·1527, 25·98, 55·68, 97·3. This case we select for numerical attack.

Alternative methods of approximate solution: (1) Finite-difference methods

230. We can obtain approximate solutions by any of four different methods, each having its analogue in one of the four methods which were reviewed in Chapter X. The finite-difference method (analogous with that of §§ 209–11) does not call for detailed discussion since it is not really an application of the relaxation technique. It substitutes in terms of finite differences for the differential coefficients in

$$\frac{d^2}{dx^2}\left(B\frac{d^2y}{dx^2}\right)+P\frac{d^2y}{dx^2} = 0, \tag{11}$$

which is the governing 'equation of neutral stability', derivable from (4) by the Calculus of Variations in the manner of § 202; and taking an assumed value of P for trial, it satisfies the resulting finite-difference equation, together with *both* of the terminal conditions imposed at one end and *one* of the terminal conditions imposed at the other, by synthesis of two solutions both relating to the same value of P. The last (i.e. fourth) of the terminal conditions will not unless by an accident be satisfied,—i.e. the solution will exhibit a finite error; but by plotting this error systematically against the related value of P, it is easy to deduce a value of P which entails no error, i.e. with which all four terminal conditions can be satisfied. This is the required 'critical end load'.

The finite-difference method seems to have been first applied to the strut problem by Bairstow and Stedman (Ref. 2): its graphical counterpart (the 'funicular' construction) was shown in 1920 (Ref. 6) to give satisfactory results. In relation to the particular problem of § 229 it was found by Bradfield (Ref. 1, Appendix II) to yield for the 'first critical load' the values

$\lambda_1 = 8{\cdot}091$, when the method was employed without 'modification',

$= 8{\cdot}167$, when the method was 'modified' in the manner of § 209 (Chap. X) to reduce the intrinsic errors of its formulae, (12)

the length of the strut being (in both instances) divided into ten parts. These values are near enough for all practical purposes to the exact value given in § 229, and closer approximation can of course be obtained by a finer subdivision (e.g. into twenty parts). On the other hand, even a like approximation is appreciably less easy to

obtain by finite-difference methods when the ends are clamped, for the reason that accurate expressions for the terminal slopes involve differences of rather high order (cf. § 211).

231. The first method used in Chapter X (§§ 201–6) is also a finite-difference method in essence, but in it the finite-difference approximations are implicit in certain formulae of approximate integration and differentiation, based on the assumption that a polynomial function can be used to represent the mode. These formulae can be applied to the strut problem, but are less suitable here than in the static deflexion problem of Chapter X, because the modes in which a strut deflects are curves not closely representable by polynomials (cf. § 234).

To illustrate this contention we shall treat the same example as before (§ 229). We know that on account of symmetry the modes in this example will fall into two classes,—the first characterized by symmetry with respect to the central point 3 and the second by 'skew symmetry' with respect to that point (which in modes of this class is not displaced). We also know from general principles that the smallest critical load will be associated with a mode of the first (symmetrical) class. Accordingly we consider the symmetrical class first, postulating that

$$y_1 - y_5 = 0, \qquad y_2 - y_4 = 0.$$

232. The terminal conditions (of simple support) require in addition that

$$y_0 = y_6 = y_0'' = y_6'' = 0, \tag{13}$$

so the first and last of the seven expressions corresponding with (25) of Chapter X reduce to

$$4104y_1 - 8235y_2 + 5080y_3 = 0, \tag{14}$$

and the other five expressions are replaced by

$$\left.\begin{aligned} 10(y_1'', y_5'') &= -108y_1 - 1080y_2 + 940y_3, \\ 10(y_2'', y_4'') &= 432y_1 - 810y_2 + 400y_3, \\ 10y_3'' &= -180y_1 + 1080y_2 - 980y_3. \end{aligned}\right\} \tag{15}$$

The expressions for $y_0', y_1',..., y_6'$ (obtainable from Table XL of Chap. X) are simplified correspondingly. They reduce to

$$\left.\begin{aligned} 120(y_0', -y_6') &= 5184y_1 - 8100y_2 + 4800y_3, \\ 120(y_1', -y_5') &= -1104y_1 + 2400y_2 - 1200y_3, \\ 120(y_2', -y_4') &= -192y_1 - 780y_2 + 960y_3, \\ y_3' &= 0. \end{aligned}\right\} \tag{16}$$

Finally the 'six-strip formulae' for V and V′ (corresponding with (3) of Chapter X) reduce in virtue of the symmetry to

$$\left.\begin{aligned} \mathbf{V} &\approx \frac{1}{840}[216\beta_1 y_1''^2+27\beta_2 y_2''^2+136\beta_3 y_3''^2], \\ \mathbf{V}' &\approx \frac{1}{840}[41y_0'^2+216y_1'^2+27y_2'^2], \\ &\text{where, according to (9) when } (1-k^2)^2 = 0{\cdot}2, \\ &\beta_1 = 0{\cdot}5690, \quad \beta_2 = 0{\cdot}8809, \quad \beta_3 = 1. \end{aligned}\right\} \tag{17}$$

Substituting in (17) from (15) and (16), and then eliminating y_3 with the aid of (14), we obtain

$$\left.\begin{aligned} 1680\mathbf{V} &= 3126556y_1^2-3801052y_1y_2+1200261y_2^2, \\ 1680\mathbf{V}' &= 13769y_1^2-14048y_1y_2+9042y_2^2, \end{aligned}\right\} \tag{18}$$

thus (in effect) reducing the order of the freedom to 2. Then the conditions for a stationary value of λ as given by (8),—namely

$$\frac{\partial \mathbf{V}}{\partial y_1} = \lambda\frac{\partial \mathbf{V}'}{\partial y_1}, \qquad \frac{\partial \mathbf{V}}{\partial y_2} = \lambda\frac{\partial \mathbf{V}'}{\partial y_2},$$

—become

$$\left.\begin{aligned} 6253111y_1-3801052y_2 &= \lambda(27539y_1-14048y_2), \\ 3801052y_1-2400521y_2 &= \lambda(14048y_1-18084y_2), \end{aligned}\right\} \tag{19}$$

whence we have on elimination of y_1, y_2

$$7{\cdot}5165\lambda^2-1809{\cdot}8\lambda+14068 = 0. \tag{20}$$

The roots of this equation are

$$\lambda_1 = 8{\cdot}042, \qquad \lambda_3 = 232{\cdot}7_5, \tag{21}$$

and corresponding values of y_2/y_1 can be obtained from (19). They are

$$(y_2/y_1)_1 = 1{\cdot}635, \qquad (y_2/y_1)_3 = -0{\cdot}2940.$$

The correct values corresponding with (21) are

$$\lambda_1 = 8{\cdot}1527, \qquad \lambda_3 = 55{\cdot}68$$

(cf. § 229), so the first critical load has been obtained with reasonable accuracy. Our estimate of λ_3, on the other hand, is so wide as to be quite useless.

233. In relation to skew-symmetrical modes the postulate of § 231 can be replaced by

$$y_1+y_5 = 0, \qquad y_2+y_4 = 0, \qquad y_3 = 0,$$

and the conditions (13) are imposed as before; so we have corresponding with (14)

$$16y_1 = 17y_2, \tag{22}$$

and everything can be expressed in terms of y_2. The order of the procedure is thus reduced (in effect) to 1, and only one value of λ will be estimated.

We have corresponding with (15)

$$\left.\begin{aligned} (y_1'', -y_5'') &= -48y_1+6y_2 = -45y_2, \\ (y_2'', -y_4'') &= 48y_1-87y_2 = -36y_2, \\ y_3'' &= 0, \end{aligned}\right\} \tag{23}$$

and corresponding with (16)

$$\left.\begin{aligned} 10(y_0', y_6') &= 288y_1-225y_2 = 81y_2, \\ 10(y_1', y_5') &= -62y_1+100y_2 = 34{\cdot}125y_2, \\ 10(y_2', y_4') &= -32y_1-5y_2 = -39y_2, \\ 10y_3' &= 18y_1-90y_2 = -70{\cdot}875y_2. \end{aligned}\right\} \tag{24}$$

The simplified formulae (17) hold as before, but they now give in place of (18)

$$1680\mathrm{V} = 559407y_2^2, \qquad 1680\mathrm{V}' = 24895y_2^2,$$

whence we have according to (8)

$$\lambda = \mathrm{V}/\mathrm{V}' = 22{\cdot}47. \tag{25}$$

The exact value is 25·98 (§ 229).

234. The discrepancy is not surprising in view of what has been said in § 231, but the sign of the error calls for comment. On Rayleigh's principle (§ 131), since we are concerned in (25) with the smallest value of λ which is compatible with skew-symmetry, an over-estimate was to be expected, whereas in fact our result is an under-estimate; and in § 232 we erred in the same direction in regard to λ_1, the smallest value of λ which is compatible with symmetry. In both instances we have worked from the conditions of a stationary value, so the discrepancies must be attributable to our approximate formulae of integration and differentiation.

It would seem that representation by polynomials (which is the basis of finite-difference methods) fails in regard to inflexional curves of the kind which occur as 'normal modes' in vibrations and in examples of elastic instability, although it appears to be satisfactory as applied to curves of static deflexion, if we may judge by the success of the first method used in Chapter X (§§ 203–8). The conclusion is

not surprising in view of remarks advanced by R. A. Frazer (Ref. 5). It suggests that finite-difference methods should be used with caution in problems of the kind considered here.

Alternative methods of approximate solution: (2) The use of a finite series of chosen functions

235. As an alternative we now proceed on the assumption (cf. § 212) that the wanted function Y can be expressed with sufficient accuracy by a finite series of chosen functions, *each of which severally fulfils all of the imposed terminal conditions.* That is, we write

$$y = Y(z) = a_1 Y_1 + a_2 Y_2 + \ldots + a_n Y_n, \tag{26}$$

where $Y_1, Y_2, \ldots, Y_n$ are the chosen functions and our problem is to determine $a_1, a_2, \ldots, a_n$. As in Chapter X we can 'relax' in two different ways,—either (*a*) 'by inspection' (when we seek values of $a_1, a_2, \ldots, a_n$ such that the governing equation holds at a number of selected points in the range $0 \leqslant z \leqslant 1$), or (*b*) 'systematically', in the manner of §§ 213–20.

The method of 'Relaxation by Inspection'

236. This method, described in relation to statical problems in §§ 221–3 of Chapter X, can be extended to cover the strut problem, and has in fact been applied by Bradfield (Ref. 3) to the example of § 229. Modifications are necessitated by the absence of 'initial loading' and the consequent indeterminacy of the solution as regards the *amplitude* of the displacement.

We shall assume in this exposition terminal conditions of simple support,† so that a suitable form for Y_r is $\sin r\pi z$. Values of Y_r and Y_r'' for $z = 0, 0{\cdot}1, \ldots, 0{\cdot}9, 1$ can be taken from Tables XLIV and XLV (at the end of this book). We integrate the equation

$$\frac{d^2}{dz^2}\{\beta(z)Y''(z)\} + \lambda Y''(z) = 0,$$

which is the form assumed by (11) when we employ the non-dimensional notation of § 229; and we say that the resulting equation

$$\beta(z)Y''(z) + \lambda Y(z) = 0 \tag{27}$$

† The device employed in §§ 221–3 to deal with clamped ends is also applicable to this problem.

expresses the equality of a disturbing moment due to thrust, given by

$$\left.\begin{aligned} \mu_T \text{ (say)} &= \lambda Y = \lambda \sum_n [a_r \sin r\pi z], \\ &\text{and of a restoring moment} \\ \mu_E \text{ (say)} &= -\beta(z)Y'' = \beta(z) \sum_n [r^2\pi^2 a_r \sin r\pi z] \end{aligned}\right\} \quad (28)$$

due to the stresses entailed by flexure.† Then the problem stated in § 235 reduces to the finding of values for a_2/a_1, a_3/a_1,..., a_n/a_1 such that μ_T and μ_E balance at each of the n sections which divide the length into $(n+1)$ equal parts; and we can derive n relations between the $(n-1)$ unknown ratios and the characteristic number λ whose value is required,—namely,

$$\pi^2\beta(z) \sum_n [r^2 a_r \sin r\pi z] = \lambda \sum_n [a_r \sin r\pi z] \quad \text{when} \quad z = (1, 2,..., n)/(n+1). \quad (29)$$

Only the *ratios* a_2/a_1,..., etc., are involved, because the absolute magnitude of Y is indeterminate. Accordingly any one of a_1, a_2,..., a_n may be made unity.

237. Giving n the value 9 (so that the numbered sections divide the length into ten equal parts, and at them z has the values 0·1, 0·2,..., 0·9), suppose that we want the first (or lowest) critical value of λ. We start by assuming that $Y = Y_1$ simply (since on general grounds we know that the first mode will be a curve of one bay), and we calculate and plot Y and $-\beta(z)Y''$ for each of the numbered sections. Then by inspection we choose a starting value for λ such that μ_T and μ_E are approximately equal and (27) is approximately satisfied.

Thereafter the relaxation process is best explained with the aid of diagrams. In each stage we either modify the mode by superposing (in suitable proportion) one of the 'group displacements' Y_2, Y_3,..., Y_n, or seek a closer balance between μ_T and μ_E by altering λ. In Fig. 39, the left-hand column shows the type of the added group displacement, the third shows the additions to μ_E and μ_T which it entails, column 2 shows the mode (Y) and column 4 the residual moment $(\mu_E-\mu_T)$ which result when a chosen amount of this group displacement has been added. Thus in line 1 the chosen group displacement is Y_1, so Y (second column) $= Y_1$ simply: values of $-\pi^2\beta(z)Y_1$ and of Y_1 are plotted above and below the line in the third column, and the

† The double differentiation of (26) is clearly permissible.

residual moment which results when λ has been chosen (and recorded) is shown in column 4. From the form of the right-hand diagram it is clear that Y_3 predominates in the required modification, and the second line relates to this particular group displacement: column 1 shows its nature, column 3 the corresponding distributions of

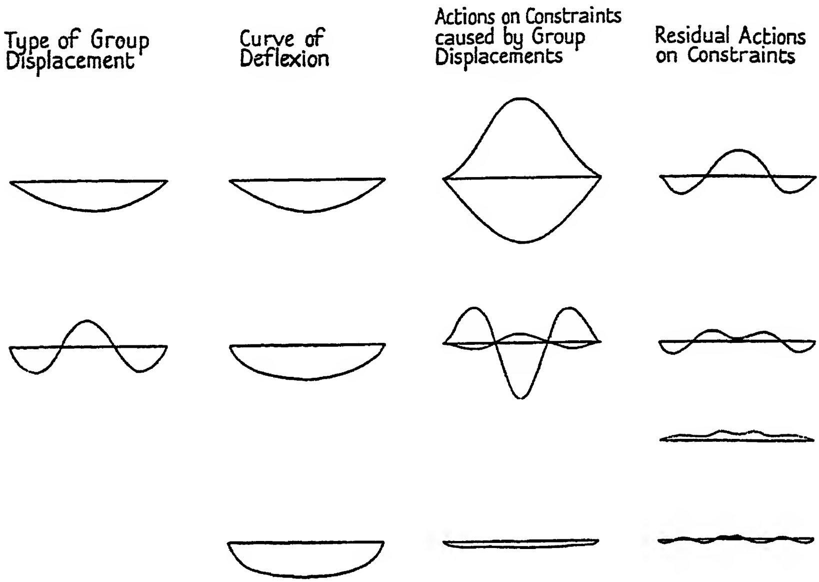

FIG. 39

$-\beta(z)Y_3''$ and of Y_3, column 2 and column 4 the mode and residual moment as modified by attaching a chosen value to a_3.

After several operations performed in this manner the residual moment will have a distribution of the kind shown in the third line, fourth column. Then it is evident (since $\mu_E - \mu_T$ is everywhere positive) that further reduction will be achieved most simply by increasing λY—i.e. by increasing λ without altering the mode. Line 4 shows the result: there is no entry in column 1 and no change in column 2, but an addition is made to λY in column 3 and the effect on the residuals is shown in column 4. Further progress would require the addition of some fairly 'high harmonic' (the diagram suggests Y_7).

238. Proceeding on these lines, in a relaxation table involving forty operations Bradfield (Ref. 3, §§ 9–10) arrived at an estimate ($\lambda_1 = 8{\cdot}16$) for the first critical load which differs by less than 0·1

per cent. from the correct value (8·1527) of § 229. Similar treatment led to estimated values 26·1, 56·5, 100·1 for the second, third and fourth critical values,—i.e. to values over-estimated by 0·5, 1·5 and 2·9 per cent.; and by subdividing the length into twenty instead of ten parts he reduced the figure 100·1 to 97·5—an over-estimate of 0·2 per cent. The starting assumptions were $Y = Y_2$, Y_3, Y_4, respectively, in these three cases: as was to be expected, Y_2 and Y_4 called for modification only by an addition of even, Y_3 (like Y_1) only by an addition of odd harmonics.

239. Thus applied to struts the method of 'Relaxation by Inspection' seems capable of giving results more than sufficiently accurate for practical purposes; but contrary to expectation less satisfactory results were obtained when the method was applied to the problem of transverse vibrations in a non-uniform rod. It was often difficult to decide the nature of the next step.

We shall turn to this problem when we have applied the last of our four alternative methods (§ 230) to the strut.

The method of 'Systematic Relaxation'

240. This is an extension of the method of §§ 213–15, Chapter X. We substitute from (26) in the expressions (8) for $\mathbf{V}$ and $\mathbf{V}'$, thus obtaining

$$\left.\begin{aligned} 2\mathbf{V} &= \int_0^1 \beta(z)(a_1 Y_1''+a_2 Y_2''+\ldots+a_n Y_n'')^2\,dz,\\ 2\mathbf{V}' &= \int_0^1 (a_1 Y_1'+a_2 Y_2'+\ldots+a_n Y_n')^2\,dz, \end{aligned}\right\} \tag{30}$$

and we employ the relaxation method to obtain a solution of the n equations typified by

$$\frac{\partial \mathbf{V}}{\partial a_r} = \lambda \frac{\partial \mathbf{V}'}{\partial a_r}, \tag{31}$$

which are the conditions for a stationary value of λ as determined from

$$\mathbf{V} = \lambda \mathbf{V}'. \tag{8) \textit{bis}$$

Only $(n-1)$ of the n equations (31) are independent, because from the set of n, multiplying by a_1, a_2,..., a_n and adding, we obtain (8) in virtue of the quadratic forms of $\mathbf{V}$ and $\mathbf{V}'$. Thus the number of independent equations is equal to the number of the unknowns,—namely, the ratios a_2/a_1, a_3/a_1,..., a_n/a_1. Because the amplitude is

indeterminate, we may keep any one of $a_1, a_2, \dots, a_n$ unity throughout the relaxation process.†

241. According to (30) the typical equation (31) may be written as

$$0 = \mathrm{A}_r = \lambda\frac{\partial \mathrm{V}'}{\partial a_r}-\frac{\partial \mathrm{V}}{\partial a_r}$$

$$= \lambda\int_0^1 (a_1 Y_1'+a_2 Y_2'+\dots+a_n Y_n')Y_r'\,dz -$$

$$-\int_0^1 \beta(z)(a_1 Y_1''+a_2 Y_2''+\dots+a_n Y_n'')Y_r''\,dz, \quad (32)$$

and we have

$$\left.\begin{aligned} \widehat{rr} &= \frac{\partial \mathrm{A}_r}{\partial a_r} = \lambda\int_0^1 Y_r'^2\,dz - \int_0^1 \beta(z) Y_r''^2\,dz \\ &= \lambda I_{rr} - J_{rr} \quad \text{(say)}, \\ \widehat{rs} &= \frac{\partial \mathrm{A}_r}{\partial a_s} = \lambda\int_0^1 Y_r' Y_s'\,dz - \int_0^1 \beta(z) Y_r'' Y_s''\,dz \\ &= \lambda I_{rs} - J_{rs} \quad \text{(say)}. \end{aligned}\right\} \quad (33)$$

In seeking to determine λ_1 (since we know that the corresponding mode will approximate to Y_1) we may keep a_1 unity and reject the first of equations (32), viz.

$$\mathrm{A}_1 = 0,$$

in accordance with § 240. Then we want to find values of $a_2, a_3, \dots, a_n$ which will make all of $\mathrm{A}_2, \dots, \mathrm{A}_n$ zero; so in the terminology of the relaxation method we may regard the a's as 'displacements', the A's as 'residual forces', and quantities of the types $\widehat{rr}$, $\widehat{rs}$ as 'influence coefficients'.

242. As in Chapter VII our problem differs from the statical problem in that the influence coefficients have values which are not constant since their expressions involve λ; but we know on the other hand (cf. § 142) that starting with a reasonably close approximation to the correct mode we shall start with a very close approximation to λ and thereafter (since we shall be working in the neighbourhood of its minimum value) λ will alter very slowly. This means that the 'influence coefficients' will not in fact alter widely as the work proceeds, and on that account they will require correction only at

† Cf. the similar argument and conclusion in §§ 130, 149 of Chap. VII.

occasional stages in the relaxation process. It is a feature of the relaxation method that in effect (§ 18) it starts afresh at every operation, treating previous results as a 'trial solution' which calls for modification: consequently we shall lose no accuracy in the final stages because in the early stages we have used influence coefficients which are not quite exact.

243. The 'Maxwell relations' are satisfied, since

$$\widehat{rs} \equiv \widehat{sr}$$

by definition; and such influence coefficients as $\widehat{rr}$ will be negative unless λ seriously over-estimates λ_1 (cf. § 142, Chap. VII). So for any one 'stage' in the relaxation process (λ being constant) the framework analogy is exact and the usual argument shows that results will converge.

The procedure for liquidation will follow Chapter VIII exactly, once we have calculated the initial forces and the influence coefficients. Starting from the assumption that $a_1 = 1$ and a_r $(r \neq 1) = 0$, we have according to (32)

$$\mathrm{A}_r = \lambda \int\limits^{1} Y_1' Y_r' \, dz - \int\limits^{1} \beta(z) Y_1'' Y_r'' \, dz$$

$$= \lambda I_{r1} - J_{r1}; \tag{34}$$

so everything turns on the calculation of I_{rr}, I_{rs}, J_{rr}, J_{rs}.

We now apply this method to the example of § 229.

Example: Strut of variable rigidity, with simply supported ends

244. Here (cf. § 236) the appropriate form for Y_r is $\sin r\pi z$. So according to (33)

$$\left.\begin{aligned} I_{rs} &= \int\limits^{1} Y_r' Y_s' \, dz = rs\pi^2 \int\limits_0^1 \cos r\pi z \cos s\pi z \, dz \\ &= 0 \quad (r \neq s), \\ \text{and} \qquad I_{rr} &= r^2\pi^2 \int\limits_0^1 \cos^2 r\pi z \, dz = \tfrac{1}{2} r^2 \pi^2. \end{aligned}\right\} \tag{35}$$

The form of function $\beta(z)$, given in (9) of § 229, entails integrals J_{rr}, J_{rs} which present no difficulty. Table LIV records values of I_{rr}, J_{rr} and J_{rs} appropriate to the first seven *odd* values of r and s;

TABLE LIV. *I and J Integrals for the Strut of* § 229

I_{rr}

$r =$	1	3	5	7	9	11	13
$I_{rr} =$	4·9348	44·413	123·37	241·81	399·72	597·11	833·98

J_{rr} *and* J_{rs}

$s =$ \ $r =$	1	3	5	7	9	11	13
1	42·281	−63·875	−21·540	−13·1388	−9·5777	−7·5836	−6·2972
3	−63·875	2,780·04	−1,851·08	−550·36	−327·36	−236·39	−188·46
5	−21·540	−1,851·08	21,209·4	−10,299·7	−2,729·55	−1,543·62	−1,092·1
7	−13·1388	−550·36	−10,299·7	81,238	−33,614	−8,230·8	−4,457·4
9	−9·5777	−327·36	−2,729·55	−33,614	221,728	−83,280·$_5$	−19,280·3
11	−7·5836	−236·39	−1,543·62	−8,230·8	−83,280·$_5$	494,492	−174,067
13	−6·2972	−188·46	−1,092·1	−4,457·4	−19,280·3	−174,067	964,302

so can be used for the estimation of symmetrical modes and the associated frequencies.

This table was calculated by D. G. Christopherson (Ref. 3), who employed it to determine λ_1, λ_3, λ_5, λ_7, λ_9. The initial forces were given by

$$\mathrm{A}_3, \mathrm{A}_5, \mathrm{A}_7, \mathrm{A}_9 = -(J_{13}, J_{15}, J_{17}, J_{19}) \tag{36}$$

according to (34) and (35), and the initial value of λ was

$$J_{11}/I_{11} = 8{\cdot}5679.$$

(Comparing with the exact value 8·1527 of § 229 we see that Rayleigh's principle yields a tolerably close approximation from the first.)

245. Next, with the aid of an operations table based on Table LIV, the initial forces (36) were liquidated until A_3, A_5 had been reduced from initial magnitudes of 63·875 and 21·540 to magnitudes less than 0·01. It was not worth while, in this preliminary stage, to relax A_7 and A_9 or to permit displacements a_7, a_9: the other 'displacements' were

$$a_3 = 0{\cdot}02953_5, \qquad a_5 = 0{\cdot}00378,$$

and for these values (with a_7 and a_9 taken as zero) new values of λ, of the residual forces, and of the influence coefficients were calculated from (8), (32) and (33). The results were

$$\lambda = 8{\cdot}1720,$$

$$\mathrm{A}_3 = -0{\cdot}521, \quad \mathrm{A}_5 = -0{\cdot}186, \quad \mathrm{A}_7 = 68{\cdot}348, \quad \mathrm{A}_9 = 29{\cdot}570,$$

with influence coefficients as given below:

TABLE LV. *Values of $\partial\mathrm{A}_r/\partial a_s$ (approximate)*

s \ r	3	5	7	9
3	−2,417·10	1,851·08	550·36	327·36
5	1,851·08	−20,201·2	10,299·7	2,729·55
7	550·36	10,299·7	−79,262	33,614
9	327·36	2,729·55	33,614	−218,461

Relaxation was then continued for a second 'stage'. At its termination the final 'forces' were

$$\mathrm{A}_3 = -0{\cdot}003, \quad \mathrm{A}_5 = 0{\cdot}000, \quad \mathrm{A}_7 = -0{\cdot}002, \quad \mathrm{A}_9 = 0{\cdot}002,$$

and the final 'displacements' were

$$a_3 = 0{\cdot}03008, \quad a_5 = 0{\cdot}00441, \quad a_7 = 0{\cdot}001079_5, \quad a_9 = 0{\cdot}000310.$$

Then using (8) again the value 8·1553 was deduced for λ, and the residual forces (to be used in starting a third 'stage') found to be

$$A_3 = -0{\cdot}008, \quad A_5 = -0{\cdot}004, \quad A_7 = -0{\cdot}019, \quad A_9 = 0{\cdot}015.$$

246. Since it was evident that further relaxation would not affect any digit in the figures for a_3, a_5, a_7, a_9, closer approximation was not attempted. The estimate 8·1553 for λ, compared with the correct value 8·1527 (§229), is only about 0·03 per cent. in error, and the estimate of the associated mode, as shown in the first section of Table LVI, agrees closely with the exact solution.

TABLE LVI. *Modes determined by 'Systematic Relaxation'*

	First mode		*Third mode*		*Fifth mode*	
$z \times 18$	*By relaxation method*	*By exact analysis*	*By relaxation method*	*By exact analysis*	*By relaxation method*	*By exact analysis*
0	0	0	0	0	0	0
1	0·1986	0·1992	0·5313	0·5415	0·6688	0·7311
2	0·3832	0·3832	0·8155	0·8155	0·5702	0·5338
3	0·5460	0·5462	0·7794	0·7660	−0·1737	−0·2212
4	0·6844	0·6846	0·5246	0·5062	−0·8263	−0·8165
5	0·7982	0·7982	0·1154	0·1108	−0·9097	−0·8936
6	0·8866	0·8866	−0·3061	−0·3103	−0·4856	−0·4676
7	0·9495	0·9496	−0·6741	−0·6727	0·1664	0·1999
8	0·9873	0·9874	−0·9170	−0·9151	0·7570	0·7756
9	1	1	−1	−1	1	1

247. It is not necessary to describe in detail the exactly similar work which led to estimates of λ_3, λ_5, λ_7, λ_9. Proceeding in the manner of Chapter VIII (with a loss of one arbitrary parameter at every advance to a 'higher mode') and ending every series of type (26) with the term involving Y_9, Christopherson obtained values for λ as shown in column 3 of Table LVII.

The figures in Table LVI give an impression of sensible error in the determination of the third and fifth modes; but Fig. 40 shows that the discrepancies are in fact small.

248. To examine the importance of including several terms in the series (26), relaxation was also effected with neglect of A_9 and a_9. The resulting estimate of λ was 8·164,—i.e. the error was increased (Table LVII, cols. 5 and 6). This result suggests that very high accuracy could be obtained, if desired, by increasing the number of the variable parameters. To put the matter to the test, λ_3 and λ_5

TABLE LVII. *Critical Loads determined by 'Systematic Relaxation'*

(1) Order of critical load	(2) Exact solution	(3) By relaxation method with five parameters	(4) Error %	(5) By relaxation method with four parameters	(6) Error %	(7) With lower modes eliminated (§ 248) and five parameters	(8) Error %
λ_1	8·153	8·155	0·025	8·164	0·14	8·153	0·025
λ_3	55·68	55·89	0·4	56·35	1·2	55·85	0·3
λ_5	150·8	155·3	3	160·6	6·5	154·0	2·1
λ_7	293·4	315·8	8	350·4	20	..	..
λ_9	484	595	24	..	..	..	..

were calculated afresh from series extending to Y_{11} and Y_{13} respectively (i.e. with the number of arbitrary parameters kept constant). As expected the accuracy was considerably improved: the last two columns of Table LVII relate to these estimates.

Dispensing with the use of the conjugate relations to eliminate (approximately) lower modes from the form investigated, Christopherson found (Ref. 3, § 23) that in this instance the accuracy was actually improved. (With the original five a's available, λ_3 and λ_5 were estimated with errors of 0·35 and 2·05 per cent. respectively.) It thus appears that this device will not always be necessary or even advisable. Evidently the question turns on the closeness with which Y_3, Y_5,..., etc., approximate to higher normal modes.

Normal vibrations of continuous systems. 'Whirling' and transverse vibrations of rods and shafts

249. An outline of vibration theory has been given in Chapter VII. It needs no modification as applied to systems which have infinite freedom (i.e. continuous systems) except that integrals replace summations in the expressions for **V** and **T** and the number of the characteristic values (natural frequencies) is infinite. Our aim in approximate computation will be to estimate a few of the smaller values and the associated modes.

As an example we now consider the free transverse vibrations of an initially straight rod. It can be shown (*Elasticity* § 502) that the governing equation is

$$\frac{d^2}{dx^2}\left(B\frac{d^2y}{dx^2}\right) = mp^2y, \tag{37}$$

where B has its usual significance, m stands for the mass per unit length, and $y\cos pt$ is the instantaneous deflexion, away from the position of equilibrium, of the section defined by x;† also that the same form of equation, with ω replacing p, governs the problem of a shaft which 'whirls' with angular velocity ω. Thus the same analysis applies to both problems.

250. If both sides of (37), multiplied by $\frac{1}{2}y$, are integrated 'by parts' over the range $0 \leqslant x \leqslant l$, then whatever be the nature of the terminal conditions, provided only that they do not permit an input or output of energy, the equation

$$\tfrac{1}{2}\int\limits_0^l B\left(\frac{d^2y}{dx^2}\right)^2 dx = \tfrac{1}{2}p^2\int\limits_0^l my^2\,dz \tag{38}$$

results. This is the energy equation

$$\mathbf{V} = p^2\mathbf{T}$$

of Chapter VII, § 128, with integral expressions for $\mathbf{V}$ and $\mathbf{T}$. It can be thrown into 'non-dimensional' form by writing

$$\begin{gathered} z = x/l, \qquad y = Y(z), \qquad B = B_0\beta(z), \\ m = m_0\mu(z); \qquad \lambda = m_0p^2l^4/B_0. \end{gathered} \tag{39}$$

Then we have

$$\left.\begin{aligned} &\mathbf{V} = \lambda\mathbf{T}, \\ \text{where}\quad &\mathbf{V} = \tfrac{1}{2}\int\limits_0^1 \beta(z)Y''^2\,dz, \qquad \mathbf{T} = \tfrac{1}{2}\int\limits_0^1 \mu(z)Y^2\,dz. \end{aligned}\right\} \tag{40}$$

A numerical example (2): Transversely vibrating rod of non-uniform rigidity

251. It is not easy to devise examples, completely soluble by exact analysis, in which neither $\beta(z)$ nor $\mu(z)$ is constant. The example which follows (suggested by the strut problem of § 229) is artificial in that flexural rigidity increasing from ends to centre is associated with a mass which decreases; but it will serve to test the accuracy of our approximate methods.

Let both ends be simply supported, and let

$$\beta(z) = \{1-4k^2z^2\}^2, \qquad \mu(z) = \{1-4k^2z^2\}^{-2}, \tag{41}$$

† $T = 2\pi/p$ is the period of a complete vibration.

z being measured from the central section. The natural frequencies of vibration are given exactly by

$$\sqrt{\lambda} = 4k^2\left[1+\frac{n^2\pi^2}{\left\{\log_e\left(\frac{1+k}{1-k}\right)\right\}^2}\right], \tag{42}$$

where λ has the significance stated in (39) and n has any integral value. The mode corresponding with any frequency λ is given by

$$\left.\begin{aligned} Y &\propto (1-4k^2z^2)^{\frac{1}{2}}\cos\left\{\mu\log_e\left(\frac{1+2kz}{1-2kz}\right)\right\} \quad \text{when } n \text{ is odd,}\\ Y &\propto (1-4k^2z^2)^{\frac{1}{2}}\sin\left\{\mu\log_e\left(\frac{1+2kz}{1-2kz}\right)\right\} \quad \text{when } n \text{ is even,}\end{aligned}\right\} \tag{43}$$

μ being a constant defined by

$$4\mu^2 = \frac{\sqrt{\lambda}}{4k^2}-1. \tag{44}$$

When $k = 0{\cdot}8$ the first four values of λ as calculated from (42) are

$$\lambda_1 = 60{\cdot}74, \qquad \lambda_2 = 552{\cdot}0, \qquad \lambda_3 = 2466, \qquad \lambda_4 = 7447. \tag{45}$$

252. Applied to this example (by Bradfield, Ref. 3, § 30) a slight modification of the method of 'relaxation by inspection' (§§ 236–7) gave an estimate 60·32 for λ_1: the fractional error of this figure is $-0{\cdot}7$ per cent. (i.e. the first natural frequency was under-estimated by 0·35 per cent.), and the associated mode was determined with an error everywhere less than 1 per cent. λ_3 was under-estimated by 1·8 per cent. (an error of only 0·9 per cent. in the frequency), but the mode was less closely estimated, the errors amounting to 8 per cent. Moreover the method was not satisfactory as applied to 'higher' modes (cf. § 239): it seemed as though the rod were trying to assume one of the configurations associated with lower frequencies.†

Our examination in this chapter has indicated that finite-difference methods are not likely to be more successful. Accordingly we shall take as our one alternative, in this example, the method of 'systematic relaxation' which in §§ 244–8 was found to give good results for the non-uniform strut.

Treatment by 'systematic relaxation'

253. The calculations (made by D. G. Christopherson: Ref. 3, § 31) followed exactly the lines of §§ 244–8, therefore do not call for

† This is the phenomenon known as 'regression to the fundamental' (Ref. 4, § 5.1).

detailed description. On account of symmetry the modes fall into two classes (cf. § 231): attention was confined to 'odd' modes (i.e. to $\lambda_1, \lambda_3, \lambda_5$), and in each instance the series taken to represent the mode was

$$Y = a_1 Y_1 + a_3 Y_3 + a_5 Y_5 + a_7 Y_7 + a_9 Y_9, \tag{46}$$

Y_r having the form $\sin r\pi z$ since both ends are 'simply supported'. The device of using the conjugate relations to eliminate lower modes (cf. § 248) was not adopted.

Y being given by (46), we have from (40)

$$\left.\begin{aligned} 2\mathbf{V} &= \int^{1} \beta(z)(a_1 Y_1'' + a_3 Y_3'' + \ldots + a_9 Y_9'')^2\, dz, \\ 2\mathbf{T} &= \int^{\bar{}} \mu(z)(a_1 Y_1 + a_3 Y_3 + \ldots + a_9 Y_9)^2\, dz, \end{aligned}\right\} \tag{47}$$

corresponding with (30) of § 240; and corresponding with (31)

$$\frac{\partial \mathbf{V}}{\partial a_r} = \lambda \frac{\partial \mathbf{T}}{\partial a_r} \quad (r = 1, 3, 5, 7, 9), \tag{48}$$

—the conditions that λ as derived from (40) may be stationary. We write

$$\begin{aligned} 0 = \mathbf{A}_r &= \lambda \frac{\partial \mathbf{T}}{\partial a_r} - \frac{\partial \mathbf{V}}{\partial a_r} \\ &= \lambda \int_0^1 \mu(z)(a_1 Y_1 + a_3 Y_3 + \ldots + a_9 Y_9) Y_r\, dz - \\ &\qquad - \int_0^1 \beta(z)(a_1 Y_1'' + a_3 Y_3'' + \ldots + a_9 Y_9'') Y_r''\, dz, \end{aligned} \tag{50}$$

corresponding with (32) of § 241, and

$$\left.\begin{aligned} \widehat{rr} = \frac{\partial \mathbf{A}_r}{\partial a_r} &= \lambda \int_0^1 \mu(z) Y_r^2\, dz - \int_0^1 \beta(z) Y_r''^2\, dz \\ &= \lambda I_{rr} - J_{rr} \quad \text{(say)}, \\ \widehat{rs} = \frac{\partial \mathbf{A}_r}{\partial a_s} &= \lambda \int_0^1 \mu(z) Y_r Y_s\, dz - \int_0^1 \beta(z) Y_r'' Y_s''\, dz \\ &= \lambda I_{rs} - J_{rs} \quad \text{(say)}, \end{aligned}\right\} \tag{51}$$

corresponding with (33). Both the I's and J's are now special to the particular example, and for that reason are not tabulated. They were calculated without difficulty, Y_r having the form $\sin r\pi z$.

TABLE LVIII. *Normal Modes and Natural Frequencies for Non-Uniform Shaft*

	First mode		*Third mode*		*Fifth mode*	
$z\times 20$	*By relaxation method*	*By exact analysis*	*By relaxation method*	*By exact analysis*	*By relaxation method*	*By exact analysis*
0	1	1	1	1	1	1
1	0·9893	0·9902	0·9346	0·9385	0·8078	0·8393
2	0·9580	0·9613	0·7423	0·7629	0·3235	0·4072
3	0·9059	0·9025	0·4446	0·4780	−0·2568	−0·1711
4	0·8328	0·8429	0·0863	0·1399	−0·7370	−0·6797
5	0·7392	0·7532	−0·2755	−0·2235	−0·9186	−0·9106
6	0·6256	0·6432	−0·5806	−0·5463	−0·6463	−0·7254
7	0·4939	0·5118	−0·7798	−0·7541	−0·0077	−0·1550
8	0·3420	0·3594	−0·7258	−0·7637	0·5545	0·4996
9	0·1755	0·1871	−0·4498	−0·5069	0·5513	0·6807
10	0	0	0	0	0	0
λ	60·96	60·74	2,589	2,466	19,840	17,795

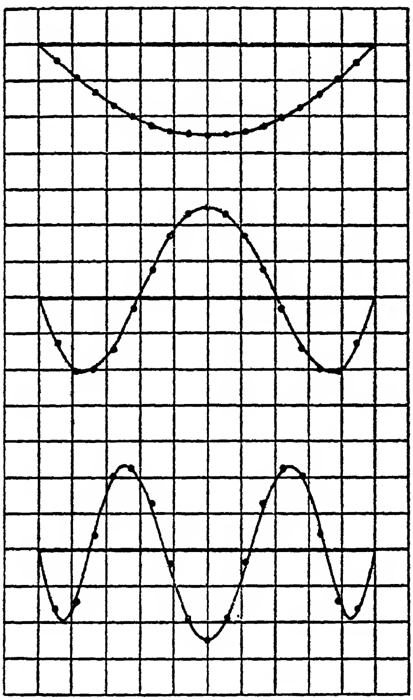

FIG. 40

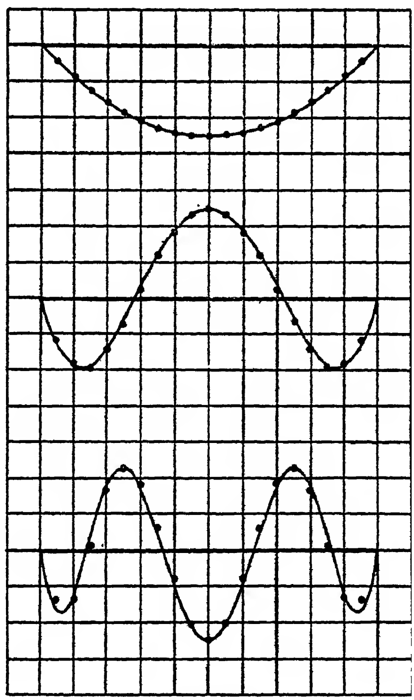

FIG. 41

254. The rest of the work followed §§ 244–8 exactly, so it will suffice to give the results. Table LVIII makes a numerical com-

parison with exact theory (§ 251). Fig. 41 serves to indicate the approximation with which modes have been determined.

The conclusion seems to be justified that Relaxation Methods provide a means of determining critical loads or natural frequencies of vibration, *together with the associated modes of distortion*, to any degree of accuracy which is likely to be wanted in practical work.

RECAPITULATION

255. Chapter X extended to continuous systems methods which had been evolved in Chapters I–VI for problems of equilibrium in systems of finite freedom. This chapter makes a similar extension of the dynamical theorems and methods of Chapters VII–IX.

Two representative problems are treated successfully: (1) critical loads for a non-uniform compression member (strut), and (2) natural frequencies and modes of vibration for a non-uniform elastic bar. Excepting §§ 231–3 (in which the computations have been done by Miss A. Pellew) almost the whole of this chapter has been taken from the paper cited in Ref. 3.

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XII

FURTHER DEVELOPMENTS. NON-LINEAR SYSTEMS. CONCLUSION

256. WITH these extensions to vibrations both forced and free, also to problems concerning the elastic stability of bars and shafts, we reach the present frontiers of the relaxation method,—regions where for a time progress must be unsystematic and advance must be attempted in whatever direction seems most likely to succeed. Broadly speaking we can cope, now, with all of those standard problems in engineering science which involve either systems of restricted freedom or systems governed by equations in one space-variable. In two dimensions also some problems have yielded to attack; but the processes of computation are so different as to justify their relegation to another volume,—especially as this field has greater interest for the physicist than for the engineer.

Problems in two dimensions

257. A bare summary of results, on the other hand, may perhaps have interest. First, Laplace's equation in two dimensions, viz.

$$\nabla^2\psi \equiv \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = 0, \tag{1}$$

can be solved (approximately) for any form of boundary, and with boundary conditions fixing either ψ or its normal gradient $\partial\psi/\partial\nu$ (Ref. 3). Physical applications of this equation are found (e.g.) in electromagnetism (it governs the distribution of electric or magnetic force), in elasticity (the torsion and flexure problems of St. Venant), and in hydrodynamics (it governs the stream-function and velocity potential of an inviscid fluid). In mathematics its most important application is to the problem of conformal transformation, which has been solved (Ref. 4) for a variety of standard cases.

258. Poisson's equation in two dimensions, viz.

$$\nabla^2 w = Z \text{ (a specified function of } x \text{ and } y), \tag{2}$$

entails precisely the same procedure; for in either problem we have in effect to liquidate 'forces imposed upon constraints' by an assumed (or trial) form of displacement, and obviously allowance can be made for Z in calculating the initial values of these forces. Equation (2)

finds application in the theory of elasticity, and the relaxation method of solution may perhaps prove useful in relation to the theory of earth pressures.

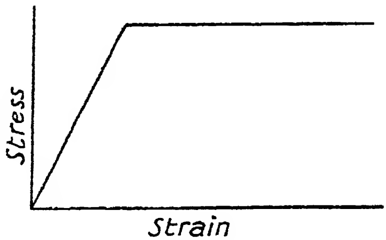

FIG. 42

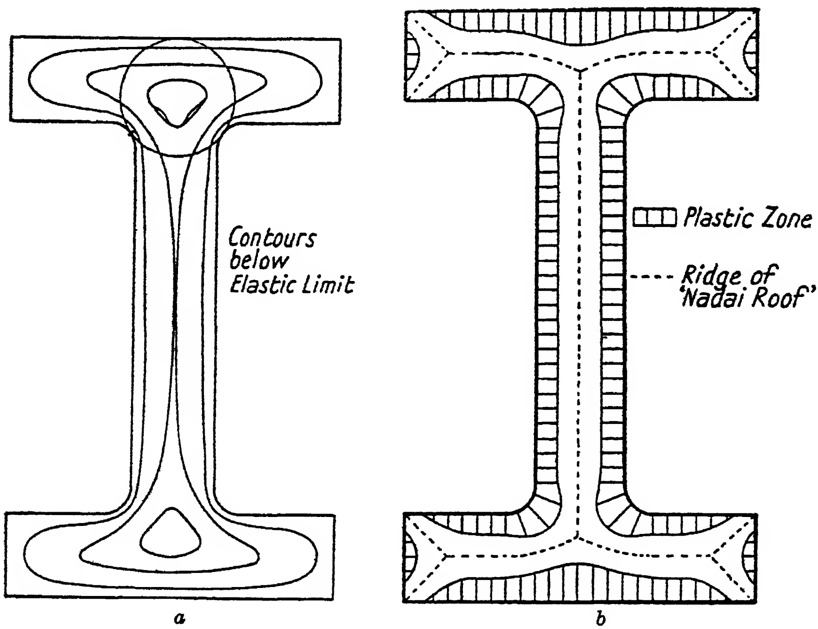

FIG. 43

A case of special interest (in that the problem appears to be literally insoluble by orthodox mathematics) is presented in the theory of plastic torsion. Assuming the relation between stress and strain to be of the type shown in Fig. 42 (i.e., linear up to some specified limit, beyond which the strain can increase indefinitely without further increase of stress), we can (Ref. 3) reduce our problem to that of solving (2) with $Z =$ const., *but under the overriding restriction that w must not exceed a specified limit at every point, so*

that (2) *is valid only in regions where this condition is satisfied.* The kind of result which follows is shown in Fig. 43 (Ref. 2, Figs. 4 and 6): When twisted by a small couple (insufficient to produce over-strain of its material), an I-section girder will be stressed as shown in Fig. 43 *a*. When on the other hand the torque is large, only its interior remains elastic and over-strain starting at its outer surface extends (with increasing torque) more and more deeply into the material, as shown in Fig. 43 *b*.

Non-linear systems

259. We have referred to this problem in some detail, because it suggests a field in which relaxation methods may have special value,—namely, that class of problems in which, since the governing equation is non-linear, *solutions cannot be superposed.* Usually, in this field, very little help is forthcoming from orthodox mathematics, consequently it is not to be expected that we shall be able to justify on theoretical grounds the processes which we have occasion to employ. But this in the outlook of the practical computer is a matter of only secondary importance: provided that he has obtained a solution of his particular problem (and this can be verified *a posteriori*), he will not be greatly concerned to know whether the same processes would have been successful in every case.

In regard to the problem of plastic torsion, on physical grounds it was evident that a unique solution must exist, and a simple modification of the standard relaxation process led, as had been expected, to its discovery. One other problem governed by non-linear equations has been attacked,† and here again success could be anticipated and was in fact realized. It relates to the stiffened suspension-bridge, and has sufficient engineering interest to justify a brief description here.

The problem of the stiffened suspension-bridge

260. According to Professor Timoshenko (Ref. 7), the theory of the stiffened suspension-bridge originated with Rankine and was developed on the assumption that the suspended truss remains absolutely rigid under the action of live load; further development

† Strictly speaking the 'characteristic value problems' of Chaps. VII–IX were non-linear, since the influence coefficients were not constant but depended on the variable which was to be determined. They were, however, *treated as linear* in each stage of the relaxation process (cf. § 142).

was made by Professor J. Melan, who first took into consideration the deflexion of the span-truss. The application of this more exact theory to the design of large American suspension-bridges, such as the Manhattan Bridge in New York and the Philadelphia–Camden Bridge, showed that it has great practical importance as permitting a considerable economy of material. In applying Melan's theory to

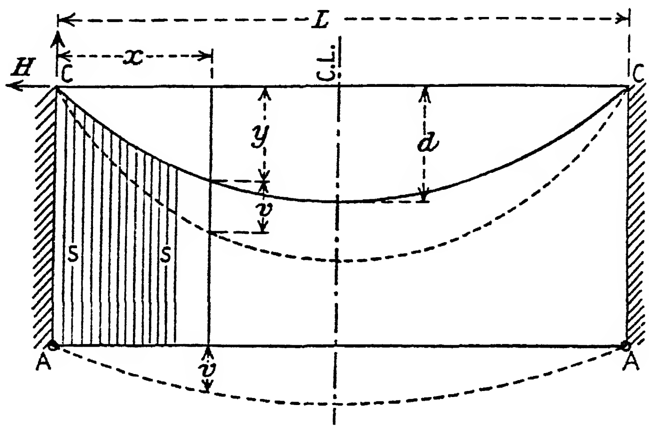

FIG. 44

determine the additional horizontal component of the cable-stress due to any cause (such as live load or temperature-change) it has often been assumed that not only dead load but also additional load may be considered as uniformly distributed along the span, and in this manner an equation has been established from which the additional cable-stress can be calculated by a 'cut-and-try' method. The assumption of uniform load-distribution is sufficiently accurate provided the span is fully loaded, or nearly so; but a considerable error may occur when only a small part of the structure is loaded.

261. With a view to developing a more precise theory, Professor Timoshenko dealt in exact terms (Ref. 7) with the simplified problem shown in Fig. 44. He assumed (in effect) that the number of the vertical suspension-rods is so great that the loading which they transfer from the girder to the cable, and hence the curve which is assumed by the latter, may be treated as continuous. The assumption is evidently legitimate in a treatment intended to elucidate principles: it reduces each truss of a suspension-bridge to the following essential components:

(*a*) an elastic girder AA, with ends constrained in some definite manner, to which the load-system is applied;
(*b*) vertical suspension-rods S, S, etc., attached to AA, whereby part of the load-system is transferred to
(*c*) a flexible cable CC, deriving its ability to sustain vertical loads from tension combined with curvature.

A truss of this kind is 'redundant', whether the ends of the girder be clamped or simply supported; for the cable-tension is not statically determinate, but may be altered by self-straining due to temperature-changes and the like. Usually the lengths of S, S, etc., are so designed that under the dead loading (w_D) the girder is practically straight and therefore devoid of stress. The problem attacked by Timoshenko was to determine what increase of tension in the cable, what loads in the suspension-rods, and what bending moments in the girder will be induced by a specified live loading (w_L). For simplicity he considered the case in which

(1) the ends of the girder are simply supported,
(2) the rods S, S, etc., are inextensible and so long that they remain sensibly vertical in the strained configuration,
(3) the rods are spaced so closely that displacements due to live load may be treated as continuous functions of the horizontal distance from one end,

and in his numerical work assumed further that the girder has uniform flexural rigidity.

262. His analysis may be summarized in the equation

$$\frac{d^2}{dx^2}\left(B\frac{d^2v}{dx^2}\right)-(H+H_L)\frac{d^2v}{dx^2}-H_L\frac{d^2y}{dx^2}=w_L, \tag{3}$$

in which on the left-hand side, of the total (specified) live load w_L, the first term gives the part sustained by the girder and the other terms give the part sustained by the cable. As indicated in Fig. 44,

v is the vertical displacement (common to cable and girder) at a section defined by x,
y is the depth of the cable at x when only dead load is operative,

also

B denotes the flexural rigidity of the girder (hereafter assumed to be uniform),

H denotes the horizontal component of the cable tension (uniform throughout the span) in the absence of live loading,

H_L denotes the increase in H which results from the imposition of live loading w_L per unit run.

What makes this a 'non-linear problem' is the occurrence in (3) of d^2v/dx^2 multiplied by H_L,—i.e. of a product of which *both* factors depend on w_L. In Timoshenko's analysis, (3) has to be solved in conjunction with a further equation relating H_L with the transverse deflexion v.

263. We do not reproduce the second equation here, for the reason that in a recent paper (Ref. 1) arguments were adduced to show that its form as given by Timoshenko is incorrect. This is not the place for detailed discussion of that question,—more especially since the alternative approach leads to equations which *from the standpoint of computation* present the same problem as Timoshenko's. It replaces equation (3) by

$$\frac{d^2}{dx^2}\left(B\frac{d^2v}{dx^2}\right)-(H+H_L)\left[\frac{d^2v}{dx^2}+\frac{d}{dx}\left(u\frac{d^2y}{dx^2}\right)\right]-H_L\frac{d^2y}{dx^2}=w_L, \qquad (4)\dagger$$

in which u as well as H_L can be calculated when v is known and when the shape of the cable under dead loading only (i.e. when the form of y) is specified.

264. In the paper cited (Ref. 1) the following procedure was employed to put the problem into a form appropriate to relaxation methods: First, the type of the live loading w_L was assumed to be specified by M_L, the corresponding bending moment as defined by

$$\left.\begin{aligned} M_L &= 0 \quad \text{when } x=0 \text{ and when } x=L,\\ \frac{d^2M_L}{dx^2}&+w_L = 0 \quad (0 \leqslant x \leqslant L). \end{aligned}\right\} \qquad (5)$$

Secondly, both v and M_L were assumed to be represented by Fourier series,‡ and the problem thus reduced to that of relating coefficients in the first series, typified by V_n, with the (known) coefficients of the second series, typified by $(M_L)_\lambda$. Finally, by somewhat lengthy

† Ref. 1, equation (11). ‡ Equations (26) and (37) of Ref. 1.

analysis which it is not necessary to reproduce, the computational problem was presented in the form of two equations,† viz.

$$\left.\begin{aligned} (M_L)_\lambda - H_L \sum_n [(A_{\lambda,n}+B_{\lambda,n})V_n] &= H \sum_n [F_{\lambda,n} V_n], \\ \text{where} \qquad H_L &= \sum_n [D_n V_n], \end{aligned}\right\} \tag{6}$$

D_n (depending on n) and $A_{\lambda,n}+B_{\lambda,n}$, $F_{\lambda,n}$ (depending both on λ and on n) being quantities which can be tabulated once for all (like influence coefficients in preceding chapters) in relation to any particular bridge.

Again the non-linearity of the problem is evident, since H_L multiplies, yet itself depends on, all the V's.

265. Computation could now be started, and the following procedure was employed (Ref. 1, § 17) to deal with this difficulty of 'non-linearity': Initially H_L was neglected in the equations typified by (6), which then reduced to linear equations relating the unknown V's with the known (M_L)'s, and could be solved by relaxation methods in the ordinary way. This first stage in computation was continued until all of the specified (M_L)'s—numbering nine in the example treated‡—had been nearly liquidated. For each operation affecting the V's the consequent alterations to H (i.e. the ΔH_L's) were recorded in an additional column of the relaxation table, and at the end of the first stage these were added to obtain H_L and hence the magnitudes of the neglected terms in (6). The correcting terms (of type $-H_L \sum_n [(A_{\lambda,n}+B_{\lambda,n})V_n]$) were then added to the unliquidated residuals of $(M_L)_1$, $(M_L)_2$,..., etc., and a second stage of liquidation was begun; then a second correction was made for H_L as above; and so on. In fact after only two stages it was found that H_L had settled to a value sufficiently constant to obviate (for practical purposes) the need of further correction; and twenty-nine operations (in all) sufficed to reduce the largest residual of type (M_L) to less than 0·02 per cent. of the largest initial value.

Wider applications of the Relaxation Method

266. This account has been given, not with the aim of describing a particular method, but in the hope that it may give the reader confidence to attempt solutions by the Relaxation Method even when

† Equations (41) and (40) of Ref. 1. ‡ Cf. Table VIII of Ref. 1.

success cannot be assured in advance. Very little computation will suffice to indicate whether the 'residuals' are in fact decreasing, and when all have been reduced to negligible values this fact will establish the solution *a posteriori*. It will still be possible, of course, that the solution is not unique; but this question, as a rule, physical intuition will enable him to decide.

The essential feature of the Relaxation Method is this, that it fixes attention *not on the quantities whose values are required, but on the quantities whose values are given*; its aim being to account for these latter quantities, not exactly, but with more and more completeness as the 'liquidation' process is continued. Thus stated, evidently it has wider application than can be established by any rigorous argument. Indeed, since it has promise of success in fields where orthodox mathematics fails completely (cf. the example of plastic torsion: § 258), it would clearly be unenterprising to restrict its use to problems in which success can be guaranteed.

Mathematical applications. Temple's extensions of the general 'relaxation theory'

267. This is not to deny that demonstration is desirable: on the contrary, no contribution could be more valuable than a rigorous examination of the question of convergence. But the aim of the method is practical, and when solutions are needed urgently the fact that rigorous proof is lacking should not be made an argument against a more pragmatical approach.

Moreover, even when theoretical convergence has been established it will still remain for examination whether *practical* convergence is attainable. (Thus many series are unsuitable for the computation of mathematical functions, although convergent in the mathematical sense; and conversely, in some instances 'semi-convergent' series are in fact employed with success, although ultimately they are divergent.) This is a further argument against over-cautious approach.

268. The ideal, of course, is convergence both practical and theoretical; practical convergence established by actual trial, and theoretical convergence established with mathematical rigour. Thanks to the work of G. Temple (Ref. 5) it may be said to have been attained in relation to most of the problems which have been treated in this book.

Our own demonstration of convergence (in Chapter V) will probably carry conviction to engineers, but will be stigmatized by mathematicians as lacking rigour: this is because the mathematician, aiming always at unqualified proof,† contemplates possibilities which on physical grounds the engineer knows to be in fact excluded.‡ Fortunately the needs of the mathematician are met by Temple's paper, which does *not* (for example) assume that convergence of the total energy to a unique minimum implies convergence of the displacements to their correct values. Most engineers would regard the implication as intuitively obvious.

269. Temple's paper goes much further than this. He has shown (§ 5) that the 'method of steepest descents' can be used to determine that group displacement which, at any given stage in the liquidation process, will be most effective; also that the relaxation method, extended in the manner of §§ 120–4, is applicable to gyrostatic systems (§ 6). He has also broken new ground, showing in his §§ 7–9 how the method can be applied to linear operational equations; in his § 10, that linear integral equations can be brought within its scope; and in his §§ 11–12, that the same is true of linear differential equations when these have been reduced to infinite systems of linear algebraic equations.

Like other references in this book to external sources, the foregoing is an altogether inadequate account of work to which the reader should devote close attention. But a full account of mathematical aspects hardly comes within the scheme of this volume, and a brief summary will have served its purpose if it attracts the attention of mathematicians to Temple's work and to the problems there mentioned as outstanding. For other failings in regard to references this author can only offer the excuse of Dr. Johnson:|| time and again it has happened that devices found necessary or convenient, which at first were thought to be of our own invention, proved later to be special applications of theorems and methods already known.††

† "The virtue of a mathematical 'proof' being rather in showing how far or when the statement 'proved' is *not* true, than in 'proving' that it *is* true" (Temple and Bickley, Ref. 6, p. 46).

‡ In much the same way, 'existence theorems' have little appeal to the engineer, who knows intuitively (e.g.) that the governing equations of a membrane, plate or shell *must* (if correctly stated) be capable of solution.

|| 'Ignorance, Madam, pure ignorance.' (Boswell's *Life*, *ann.* 1755.)

†† Thus the relaxation process as applied to 'normalized' equations (§ 124) is in essence identical with 'Seidel's process' (Ref. 8, § 130).

Conclusion

270. Starting from one simple notion in regard to frameworks—that the actions in the members should be determinable by a calculation of which every step is the theoretical counterpart of a physical operation that can be visualized—the work described in this volume has aimed consistently at greater accuracy and wider scope. Throughout the standpoint has been adopted that a computational method, to be really satisfactory, on the one hand must demand no more of the actual computer than a knowledge of the first four rules of arithmetic together with ability to use mathematical tables, on the other must call for no mechanical aids more cumbrous than a 20-inch slide rule or (when special accuracy is required) some standard make of calculating machine.

This book presents the results of some five years' inquiry, as these relate to problems which concern the practical engineer. Further results, as having physical rather than engineering interest, have been merely summarized in this final chapter. Most of the problems which have been taken as examples relate to structural engineering, but the mechanical engineer should find value in the new attack on vibrational problems, and some electrical applications have been indicated. The bias, such as it is, can be attributed to personal predilection: what matters is that the problems should be regarded as merely illustrative, the Relaxation Method itself as completely general.

Though calculation is not the whole business of the designer, it is a very essential part. Only experience can enable him to formulate his problems, but once formulated they must be solved, and the way to a solution should not be obstructed by difficulties which have no physical counterpart. He does not want exact methods, for his problems cannot be formulated with precision: generality is what matters, not the last refinement in accuracy. To men that have this outlook and these needs, Relaxation Methods are offered as a 'new mathematics' more powerful though less rigorous than the old.

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TABLE XXXIX. *'Seven-point' Formulae of Differentiation* (cf. § 197)

The table gives values of $A_0, A_1, A_2,..., A_6$ appropriate to the mth differential of y ($m = 1, 2,..., 6$) at the point $x/l = r/6$ ($r = 0, 1, 2,..., 6$), for insertion in the formula

$$\frac{h^m}{m!} y_r^{(m)} \approx \frac{1}{6!}(A_0 y_0 + A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_4 + A_5 y_5 + A_6 y_6).$$

m	r	A_0	A_1	A_2	A_3	A_4	A_5	A_6	E
1	0	−1,764	4,320	−5,400	4,800	−2,700	864	−120	$h^7 y^{vii} \times$ 1/7
	1	−120	−924	1,800	−1,200	600	−180	24	−1/42
	2	24	−288	−420	960	−360	96	−12	1/105
	3	−12	108	−540	0	540	−108	12	−1/140
	4	12	−96	360	−960	420	288	−24	1/105
	5	−24	180	−600	1,200	−1,800	924	120	−1/42
	6	120	−864	2,700	−4,800	5,400	−4,320	1,764	1/7
2	0	1,624	−6,264	10,530	−10,160	5,940	−1,944	274	$h^7 y^{vii} \times$ −7/20
	1	274	−294	−510	940	−570	186	−26	11/360
	2	−26	456	−840	400	30	−24	4	−1/180
	3	4	−54	540	−980	540	−54	4	$h^8 y^{viii} \times$ −1/1120
	4	4	−24	30	400	−840	456	−26	$h^7 y^{vii} \times$ 1/180
	5	−26	186	−570	940	−510	−294	274	−11/360
	6	274	−1,944	5,940	−10,160	10,530	−6,264	1,624	7/20

3	0	−735	3,480	−6,915	7,440	−4,605	1,560	−225	$h^7y^{vii} \times$ 29/90
	1	−225	840	−1,245	960	−435	120	−15	7/720
	2	−15	−120	525	−720	435	−120	15	−1/90
	3	15	−120	195	0	−195	120	−15	−7/720
	4	−15	120	−435	720	−525	120	15	−1/90
	5	15	−120	435	−960	1,245	−840	225	7/720
	6	225	−1,560	4,605	−7,440	6,915	−3,480	735	29/90
4	0	175	−930	2,055	−2,420	1,605	−570	85	$h^7y^{vii} \times$ −7/48
	1	85	−420	855	−920	555	−180	25	−1/36
	2	25	−90	105	−20	−45	30	−5	1/144
	3	−5	60	−195	280	−195	60	−5	$h^8y^{viii} \times$ 7/5760
	4	−5	30	−45	−20	105	−90	25	$h^7y^{vii} \times$ −1/144
	5	25	−180	555	−920	855	−420	85	1/36
	6	85	−570	1,605	−2,420	2,055	−930	175	7/48
5	0	−21	120	−285	360	−255	96	−15	$h^7y^{vii} \times$ 5/144
	1	−15	84	−195	240	−165	60	−9	1/72
	2	−9	48	−105	120	−75	24	−3	1/720
	3	−3	12	−15	0	15	−12	3	−1/360
	4	3	−24	75	−120	105	−48	9	1/720
	5	9	−60	165	−240	195	−84	15	1/72
	6	15	−96	255	−360	285	−120	21	5/144
6	0								$h^7y^{vii} \times$ −1/240
	1								−1/360
	2								−1/720
	3	1	−6	15	−20	15	−6	1	$h^8y^{viii} \times$ −1/2880
	4								$h^7y^{vii} \times$ 1/720
	5								1/360
	6								1/240

TABLE XLIV. $Y_r = \sin r\pi z$. (*Type-solutions for terminal conditions of 'simple support'*)

z	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
0	0	0	0	0	0	0	0	0	0
0·1	0·3090170	0·5877853	0·8090170	0·9510565	1·00	0·9510565	0·8090170	0·5877853	0·3090170
0·2	0·5877853	0·9510565	0·9510565	0·5877853	0	−0·5877853	−0·9510565	−0·9510565	−0·5877853
0·3	0·8090170	0·9510565	0·3090170	−0·5877853	−1·00	−0·5877853	0·3090170	0·9510565	0·8090170
0·4	0·9510565	0·5877853	−0·5877853	−0·9510565	0	0·9510565	0·5877853	−0·5877853	−0·9510565
0·5	1·0000000	0	−1·0000000	0	1·00	0	−1·0000000	0	1·0000000
0·6	0·9510565	−0·5877853	−0·5877853	0·9510565	0	−0·9510565	0·5877853	0·5877853	−0·9510565
0·7	0·8090170	−0·9510565	0·3090170	0·5877853	−1·00	0·5877853	0·3090170	−0·9510565	0·8090170
0·8	0·5877853	−0·9510565	0·9510565	−0·5877853	0	0·5877853	−0·9510565	0·9510565	−0·5877853
0·9	0·3090170	−0·5877853	0·8090170	−0·9510565	1·00	−0·9510565	0·8090170	−0·5877853	0·3090170
1·0	0	0	0	0	0	0	0	0	0
$\sum_0^{1\cdot0}(Y)$	6·3137516	0	1·9626104	0	1·00	0	0·5095256	0	0·1583844
$\vert\sum_0^{1\cdot0}\vert(Y)$	6·3137516	6·1553672	6·3137516	6·1553672	5·00	6·1553672	6·3137516	6·1553672	6·3137516

TABLE XLV. $Y_r'' = -r^2\pi^2 \sin r\pi z$. (*Type-solutions for terminal conditions of 'simple support'*)

z	Y_1''	Y_2''	Y_3''	Y_4''	Y_5''	Y_6''	Y_7''	Y_8''	Y_9''
0	0	0	0	0	0	0	0	0	0
0·1	−3·0498755	−23·204834	−71·862100	−150·184823	−246·74011	−337·91585	−391·24921	−371·27734	−247·03992
0·2	−5·8012084	−37·546206	−84·478963	−92·819334	0	208·84350	459·94102	600·73929	469·89788
0·3	−7·9846777	−37·546206	−27·448880	92·819334	246·74011	208·84350	−149·44390	−600·73929	−646·75890
0·4	−9·3865514	−23·204834	52·210875	150·184823	0	−337·91585	−284·25921	371·27734	760·31066
0·5	−9·8696044	0	88·826440	0	−246·74011	0	483·61062	0	−799·43796
0·6	−9·3865514	23·204834	52·210875	−150·184823	0	337·91585	−284·25921	−371·27734	760·31066
0·7	−7·9846777	37·546206	−27·448880	−92·819334	246·74011	−208·84350	−149·44390	600·73929	−646·75890
0·8	−5·8012084	37·546206	−84·478963	92·819334	0	−208·84350	459·94102	−600·73929	469·89788
0·9	−3·0498755	23·204834	−71·862100	150·184823	−246·74011	337·91585	−391·24921	371·27734	−247·03992
1·0	0	0	0	0	0	0	0	0	0
$\sum_0^{1\cdot0}(Y)$	−62·3142304	0	−174·331696	0	−246·74011	0	−246·41198	0	−126·61852
$\vert\sum_0^{1\cdot0}\vert(Y)$	62·3142304	243·004160	560·828076	972·016628	1,233·70055	2,187·03740	3,053·39730	3,888·06652	5,047·45268

TABLE XLVI. $Y_r = \sin r\pi z + r\pi\{-z+(2+\cos r\pi)z^2-(1+\cos r\pi)z^3\}$. (*Terminal conditions of 'clamping'*)

z	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
0	0	0	0	0	0	0	0	0	0
0·1	0·0262737	0·1353960	−0·0392130	0·0462778	−0·4137167	−0·4061115	−1·1701864	−1·2217721	−2·2356730
0·2	0·0851305	0·3478707	−0·5569080	−0·6185863	−2·5132741	−2·3973427	−4·4696403	−3·3637997	−5·1116787
0·3	0·1492825	0·4232689	−1·6701864	−1·6433604	−4·2986723	−2·1711480	−4·3091242	−1·1600938	−5·1285931
0·4	0·1970743	0·2861924	−2·8497320	−1·5542423	−3·7699112	0·0462778	−4·6900904	−1·7941569	−7·7368966
0·5	0·2146018	0	−3·3561945	0	−2·9269908	0	−6·4977871	0	−6·0685835
0·6	0·1970743	−0·2861924	−2·8497320	1·5542423	−3·7699112	−0·0462778	−4·6900904	1·7941569	−7·7368966
0·7	0·1492825	−0·4232689	−1·6701864	1·6433604	−4·2986723	2·1711480	−4·3091242	1·1600938	−5·1285931
0·8	0·0851305	−0·3478707	−0·5569080	0·6185863	−2·5132741	2·3973427	−4·4696403	3·3637997	−5·1116787
0·9	0·0262737	−0·1353960	−0·0392130	−0·0462778	−0·4137167	0·4061115	−1·1701864	1·2217721	−2·2356730
1·0	0	0	0	0	0	0	0	0	0
$\sum_0^{1·0}(Y)$	1·1301238	0	−13·5882733	0	−24·9181394	0	−35·7758697	0	−46·4942663

TABLE XLVII. $Y_r'' = -r^2\pi^2 \sin r\pi z + 2r\pi\{2+\cos r\pi-3(1+\cos r\pi)z\}$. (*Terminal conditions of 'clamping'*)

z	Y_1''	Y_2''	Y_3''	Y_4''	Y_5''	Y_6''	Y_7''	Y_8''	Y_9''
0	6·28319	37·69911	18·84956	75·39822	31·41593	113·09734	43·98230	150·79645	56·54867
0·1	3·23331	6·95446	−53·01254	−89·86624	−215·32418	−247·43798	−347·26691	−250·64018	−190·49125
0·2	0·48198	−14·92674	−65·62940	−47·58040	31·41593	276·70190	503·92332	691·21716	526·44655
0·3	−1·70149	−22·46657	−8·59932	122·97862	278·15604	254·08243	−105·46160	−540·42071	−590·21023
0·4	−3·10336	−15·66501	71·06044	165·26446	31·41593	−315·29638	−240·27691	401·43663	816·85933
0·5	−3·58641	0	107·67600	0	−215·32418	0	527·59292	0	−742·88929
0·6	−3·10336	15·66501	71·06044	−165·26446	31·41593	315·29638	−240·27691	−401·43663	816·85933
0·7	−1·70149	22·46657	−8·59932	−122·97862	278·15604	−254·08243	−105·46160	540·42071	−590·21023
0·8	0·48198	14·92674	−65·62940	47·58040	31·41593	−276·70190	503·92332	−691·21716	526·44655
0·9	3·23331	−6·95446	−53·01254	89·86624	−215·32418	247·43798	−347·26691	250·64018	−190·49125
1·0	6·28319	−37·69911	18·84956	−75·39822	31·41593	−113·09734	43·98230	−150·79645	56·54867

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